Capacity Pre-log of Noncoherent SIMO Channels via Hironaka's Theorem

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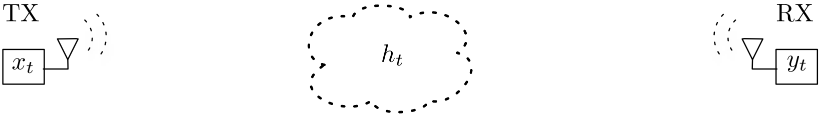
Joint work with
E. Riegler, W. Yang, G. Durisi, S. Lin, B. Sturmfels, and H. Bőlcskei

SISO Fading Channel



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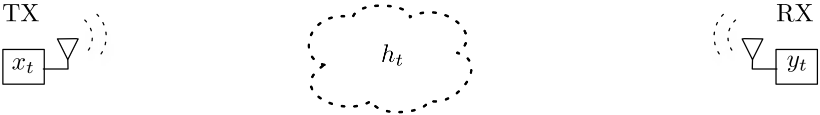

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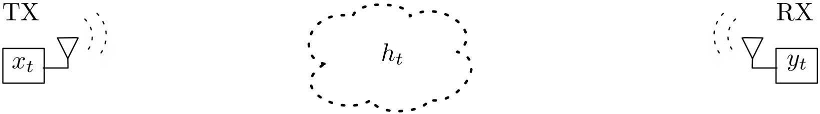

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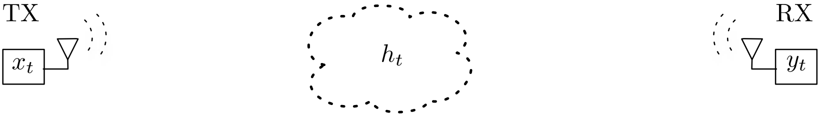

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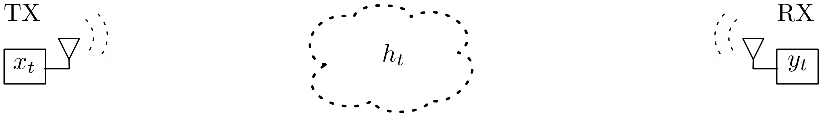

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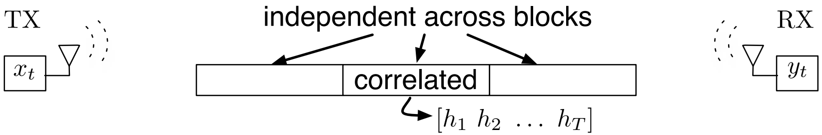

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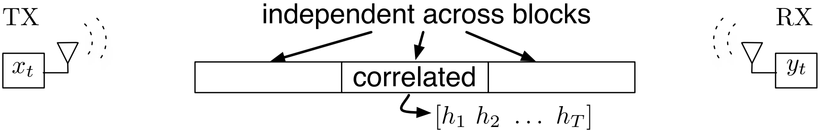

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- $[h_1 \cdots h_T]^{\mathsf{T}} \sim \mathcal{CN}(\mathbf{0}, \mathbf{P}\mathbf{P}^{\mathsf{H}})$
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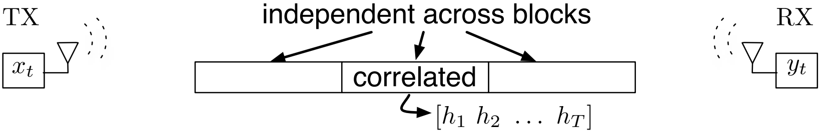

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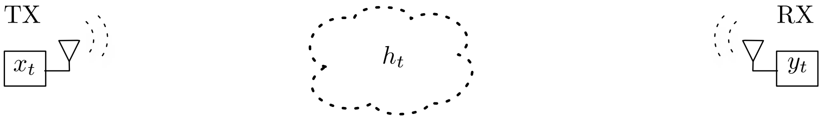
- Channel gains in whitened form:

$$\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_T \end{bmatrix} = \mathbf{P} \begin{bmatrix} s_1 \\ \vdots \\ s_Q \end{bmatrix}, \qquad s_q \sim \mathcal{CN}(0,1),\ \text{iid across } q$$

Capacity



Fundamental problem:

What is the ultimate limit on the rate of reliable communication over the channel — capacity?

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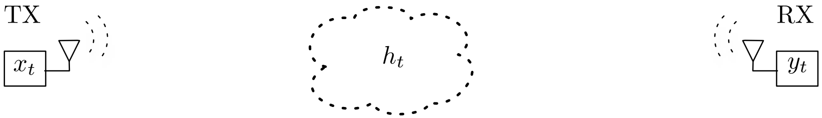

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Shannon coding theorem:

$$C(\text{SNR}) \triangleq (1/T) \sup_{f_{\mathbf{x}}(\cdot)} I(\{x_t\};\{y_t\}) \,; \qquad \mathbb{E}\left[\sum_{t=1}^{T} |x_t|^2\right] \leq T$$

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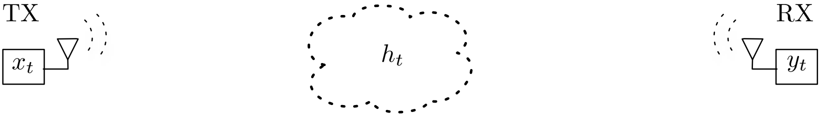

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- Mutual information: $I(\{x_t\};\{y_t\}) = h(\{x_t\}) - h(\{x_t\} \mid \{y_t\})$
- Differential Entropy: $h(\{y_t\})$

Capacity Pre-log

- AWGN Channel ($h_t = 1 \ \forall t$) [Shannon, 48]:

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Pre-Log

$$\chi = \lim_{\text{SNR} \to \infty} \frac{C(\text{SNR})}{\log(\text{SNR})}$$

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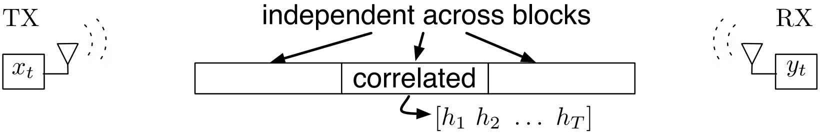

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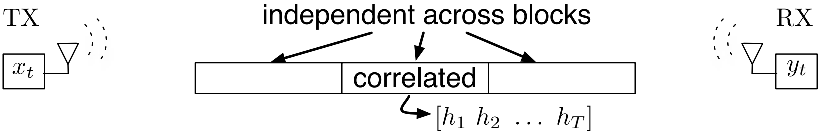
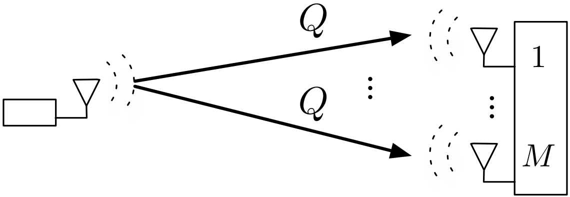


Number of unknown channel parameters per block is $Q = \operatorname{rank}\mathbf{P}$:

$$[h_1\ h_2\ \ldots h_T]^{\mathsf{T}} = \mathbf{P}[s_1\ s_2\ \ldots s_Q]^{\mathsf{T}}$$

Eliminating the Noncoherent Penalty by Adding Receive Antennas (Only!)

The Noncoherent Penalty in the SIMO Case



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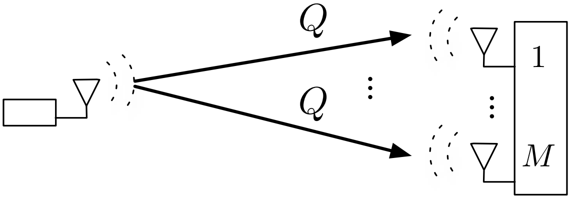

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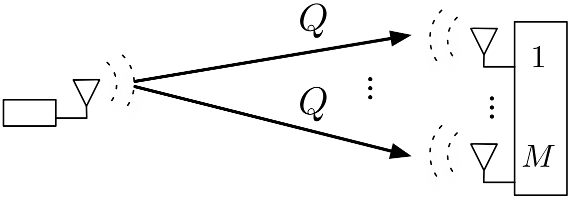

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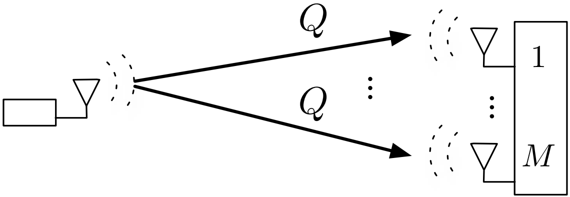

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We can actually do better!

Main Result [VM et. al., 2012]

Under technical conditions on $\mathbf{P}$,

$$\chi_{\mathsf{SIMO}} = \min[1 - 1/T, R(1 - Q/T)]$$

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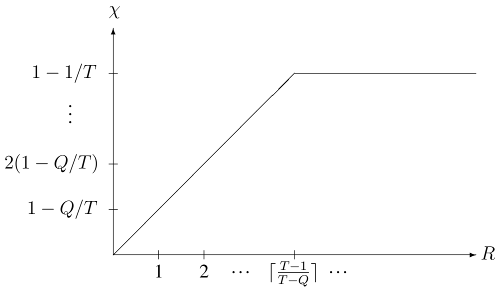

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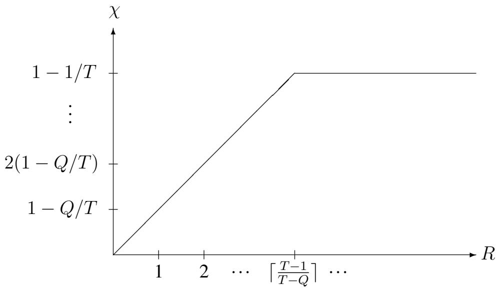

Multiple antennas at the receiver can recover degrees of freedom

Implications

$M = 2, T \geq 2Q - 1$

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The "channel identification penalty" vanishes if a second receive antenna is used

Linear Algebra: Guessing Pre-Log

Guessing the Pre-Log

Noiseless I/O relation:

$$\hat{y}_{mt} = h_{mt}x_t$$

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Rule of thumb:

$$\chi = \frac{\text{number of RVs } x_t \text{ that can be identified uniquely from } \{\hat{y}_{mt}\}}{T}$$

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- $\Rightarrow \chi_{\mathsf{SISO}} = 3/3 = 1$ [Telatar, 99]

Noncoherent SISO Channel $T = 3,\ Q = 2$

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}_{\mathbf{P}} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

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$$1\hat{y}_1 = s_1 \qquad 1\hat{y}_2 = s_2 \qquad z_3 \hat{y}_3 = (s_1 + s_2)$$

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- $\Rightarrow \chi_{\text{SISO}} = (3 - 2)/3 = 1/3 = 1 - Q/T$ [Liang & Veeravalli, 04]

Noncoherent SIMO Channel $(T = 3,\ Q = 2,\ M = 2)$

$$\hat{y}_{11} = s_{11}x_1$$

$$\hat{y}_{12} = s_{12}x_2$$

$$\hat{y}_{13} = (s_{11} + s_{12})x_3$$

$$\hat{y}_{21} = s_{21}x_1$$

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Noncoherent SIMO Channel $(T = 3,\ Q = 2,\ M = 2)$

$$
\begin{aligned}
\hat{y}_{11} &= s_{11} x_1 & z_1 \hat{y}_{11} &= s_{11} \\
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- Quadratic equations! Define $z_i = 1/x_i$

Noncoherent SIMO Channel $(T = 3,\ Q = 2,\ M = 2)$

$$\hat{y}_{11} = s_{11}x_1 \qquad z_1\hat{y}_{11} = s_{11}$$

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- 6 linear equations in 7 unknowns: $z_1, z_2, z_3, s_{11}, s_{12}, s_{21}, s_{22}$

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  $\Rightarrow$ infinitely many solutions

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- $\Rightarrow \chi_{\mathsf{SIMO}} = (3-1)/3 = 2/3 = 1 - 1/T > 1/3 = \chi_{\mathsf{SISO}}$

Equations for General $\mathbf{P}$ $(T = 3,\ Q = 2,\ M = 2)$

$$\underbrace{\begin{bmatrix} p_{11} & p_{12} & 0 & 0 & 0 & 0 \\ p_{21} & p_{22} & 0 & 0 & \hat{y}_{12} & 0 \\ p_{31} & p_{32} & 0 & 0 & 0 & \hat{y}_{13} \\ 0 & 0 & p_{11} & p_{12} & 0 & 0 \\ 0 & 0 & p_{21} & p_{22} & \hat{y}_{22} & 0 \\ 0 & 0 & p_{31} & p_{32} & 0 & \hat{y}_{23} \end{bmatrix}}_{\mathbf{B}} \begin{bmatrix} s_{11} \\ s_{12} \\ s_{21} \\ s_{22} \\ -z_2 \\ -z_3 \end{bmatrix} = \begin{bmatrix} \hat{y}_{11} \\ 0 \\ 0 \\ \hat{y}_{21} \\ 0 \\ 0 \end{bmatrix}$$

Solution is unique iff $\mathbf{B}$ is full-rank

Information Theory

Vector Notations ($T = 3,\ Q = 2,\ M = 2$)

$$\underbrace{\begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{23} \end{bmatrix}}_{\mathbf{y}} = \sqrt{\text{SNR}} \underbrace{\begin{bmatrix} s_{11}x_1 \\ s_{12}x_2 \\ (s_{11}+s_{12})x_3 \\ s_{21}x_1 \\ s_{22}x_2 \\ (s_{21}+s_{22})x_3 \end{bmatrix}}_{\hat{\mathbf{y}}} + \underbrace{\begin{bmatrix} w_{11} \\ w_{12} \\ w_{13} \\ w_{21} \\ w_{22} \\ w_{23} \end{bmatrix}}_{\mathbf{w}}$$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{s} = [s_{11}\ s_{12}\ s_{21}\ s_{22}]^{\mathsf{T}}$$

$$\mathbf{x} = [x_1\ x_2\ x_3]^{\mathsf{T}}$$

The Lower Bound $(T = 3,\ Q = 2,\ M = 2)$

- Transmit $x_i \sim \mathcal{CN}(0, 1)$, i.i.d.

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$$\begin{aligned} h(\mathbf{y}) &= h\big(\sqrt{\textsc{snr}}\hat{\mathbf{y}} + \mathbf{w}\big) \\ &\geq h\big(\sqrt{\textsc{snr}}\hat{\mathbf{y}} + \mathbf{w} \mid \mathbf{w}\big) \end{aligned}$$

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-

$$
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$$\chi_{\text{SIMO}} \geq (6 - 4)/3 = 2/3$$

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$$\chi_{\mathsf{SIMO}} \geq (6-4)/3 = 2/3 > 1/3 = \chi_{\mathsf{SISO}}$$

Is $h(\hat{\mathbf{y}})$ Finite? Change of Variables

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Change of Variables Lemma:

$$h(\hat{\mathbf{y}} \,|\, x_1) = h(\mathbf{s}, x_2, x_3 \,|\, x_1) + \mathbb{E}_{\mathbf{s},\mathbf{x}} \log \left| \det \frac{\partial \hat{\mathbf{y}}}{\partial(\mathbf{s}, x_2, x_3)} \right|$$

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Change of Variables Lemma:

$$h(\hat{\mathbf{y}} \mid x_1) = \underbrace{h(\mathbf{s}, x_2, x_3 \mid x_1)}_{\text{finite!}} + \underbrace{2\,\mathbb{E}_{\mathbf{s},\mathbf{x}} \log \left| \det \frac{\partial \hat{\mathbf{y}}}{\partial(\mathbf{s}, x_2, x_3)} \right|}_{\text{finite?}}$$

Resolution of Singularities

How can we show that

$$\mathbb{E}_{\mathbf{s},\mathbf{x}} \log \left| \det \frac{\partial \hat{\mathbf{y}}}{\partial(\mathbf{s}, x_2, x_3)} \right| > -\infty?$$

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- $J_1(\mathbf{x})$ and $J_3(\mathbf{x})$ are diagonal matrices
- $\det J_2(\mathbf{s})$ is a homogeneous polynomial:

$$\det J_2(\lambda \mathbf{s}) = \lambda^D \det J_2(\mathbf{s}), \ \forall \lambda \in \mathbb{C}$$

Resolution of Singularities (Cont'd)

Polar coordinates: $\mathbf{s} \to (r, \theta)$

$$\begin{aligned}
&\left| \int_{\mathbb{C}^{RQ}} \exp(-\|\mathbf{s}\|^2) \log |\det J_2(\mathbf{s})| \, d\mathbf{s} \right| \\
&\leq \left| \int_{\mathbb{C}^{RQ}} \exp(-\|s\|^2) \log \left|\det J_2(\mathbf{s}/\|s\|^2)\right| d\mathbf{s} \right| + O(1) \\
&\leq \int_0^{\infty} \exp(-r^2) r^{2D-1} dr \times \int_{\Delta} |\log |f(\theta)|| \, d\theta + O(1)
\end{aligned}$$

where

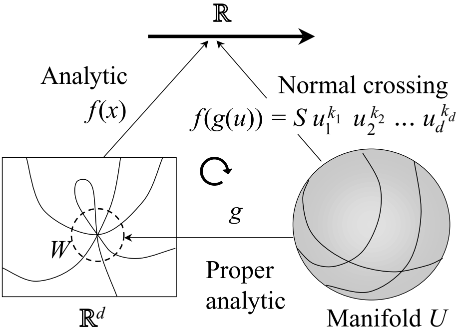
- $\Delta = [0, \pi]^{2D-2} \times [0, 2\pi]$ is a compact set
- f is a real analytic function

Resolution of Singularities (Cont'd)

Hironaka's Theorem implies:

If $f \not\equiv 0$ is a real analytic function, then

$$\int_\Delta |\log|f(\theta)||\, d\theta < \infty.$$



The Technical Condition on $\mathbf{P}$: not Just Rank

$$f \not\equiv 0 \text{ iff } \begin{bmatrix} p_{11} & p_{12} & 0 & 0 & 0 & 0 \\ p_{21} & p_{22} & 0 & 0 & p_{21} & 0 \\ p_{31} & p_{32} & 0 & 0 & 0 & p_{31} \\ 0 & 0 & p_{11} & p_{12} & 0 & 0 \\ 0 & 0 & p_{21} & p_{22} & p_{22} & 0 \\ 0 & 0 & p_{31} & p_{32} & 0 & p_{32} \end{bmatrix} \text{ is full-rank}$$

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For $T = 3,\ Q = 2,\ M = 2$ equivalent to:

Every two rows of $\mathbf{P}$ are linearly independent

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For $T = 3,\ Q = 2,\ M = 2$ equivalent to:

Every two rows of $\mathbf{P}$ are linearly independent

For example, $\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ satisfies this condition

Connection to Linear Algebra

$$\begin{bmatrix} p_{11} & p_{12} & 0 & 0 & 0 & 0 \\ p_{21} & p_{22} & 0 & 0 & p_{21} & 0 \\ p_{31} & p_{32} & 0 & 0 & 0 & p_{31} \\ 0 & 0 & p_{11} & p_{12} & 0 & 0 \\ 0 & 0 & p_{21} & p_{22} & p_{22} & 0 \\ 0 & 0 & p_{31} & p_{32} & 0 & p_{32} \end{bmatrix} \text{ is full-rank}$$

$$\text{iff } \begin{bmatrix} p_{11} & p_{12} & 0 & 0 & 0 & 0 \\ p_{21} & p_{22} & 0 & 0 & \hat{y}_{12} & 0 \\ p_{31} & p_{32} & 0 & 0 & 0 & \hat{y}_{13} \\ 0 & 0 & p_{11} & p_{12} & 0 & 0 \\ 0 & 0 & p_{21} & p_{22} & \hat{y}_{22} & 0 \\ 0 & 0 & p_{31} & p_{32} & 0 & \hat{y}_{23} \end{bmatrix} = \mathbf{B} \text{ is full-rank a.e.}$$

Open problems

- MIMO
- Stationary Channel Model

Thank you