“Crystallization” in Large Fading Networks

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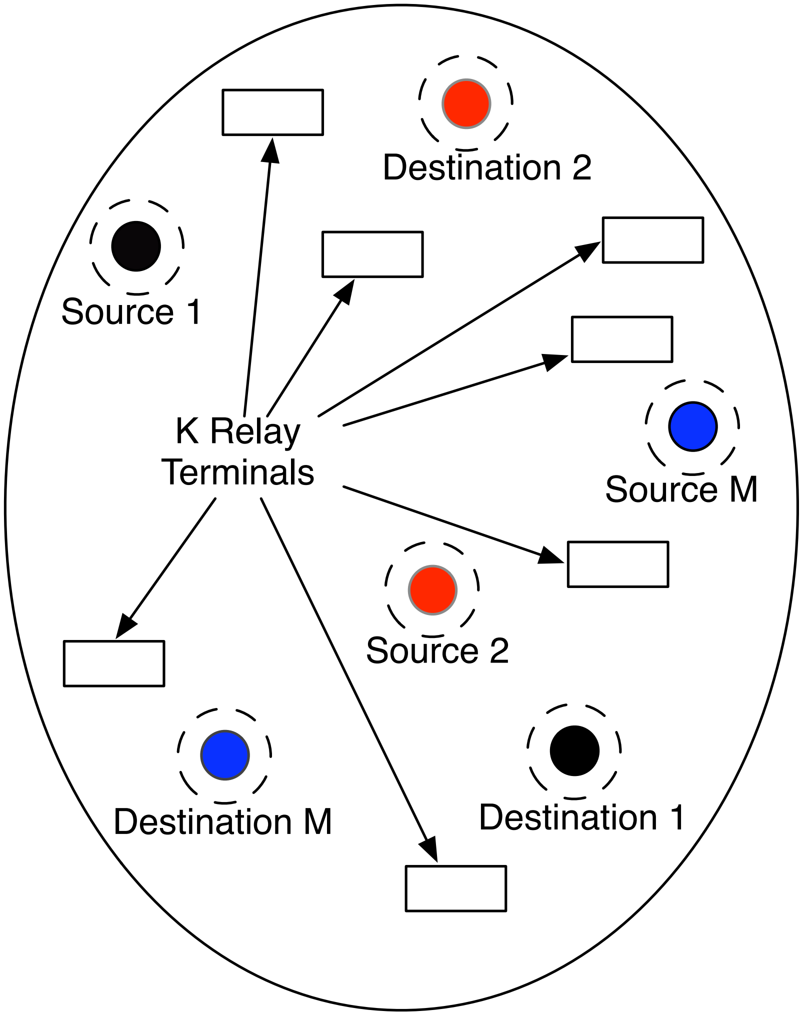
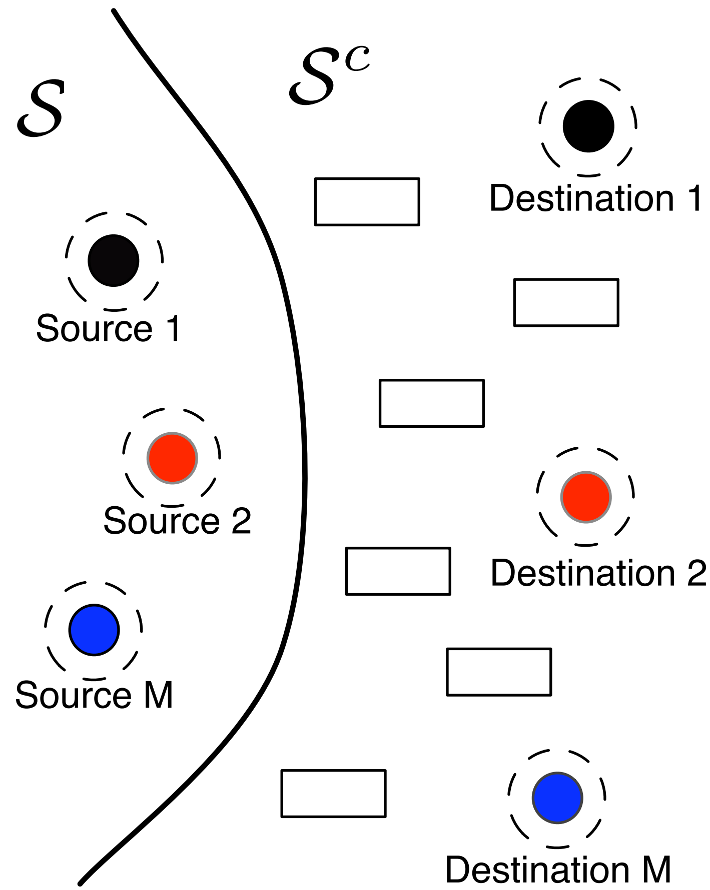
joint work with Helmut Bölcskei

An Interference Network with Relays



- **Relays** have no traffic requirements
- **No direct links** between sources and destinations
- **Single-antenna** transceivers
- **No cooperation** between sources and between destinations
- **Zone free of relays** around each source and destination
- **Bounded area**

Cut-Set Upper Bound on Network Capacity



- Max-flow min-cut theorem

$$\sum_{i \in \mathcal{S}, j \in \mathcal{S}^c} R^{(i,j)} \leq I\left(\mathbf{X}^{(S)}; \mathbf{Y}^{(S^c)} | \mathbf{X}^{(S^c)}\right)$$

yields (for large K)

$$C \leq \frac{M}{2} \log(K) + O(1)$$

- This **bound is achieved** with **cooperation** in a MIMO system

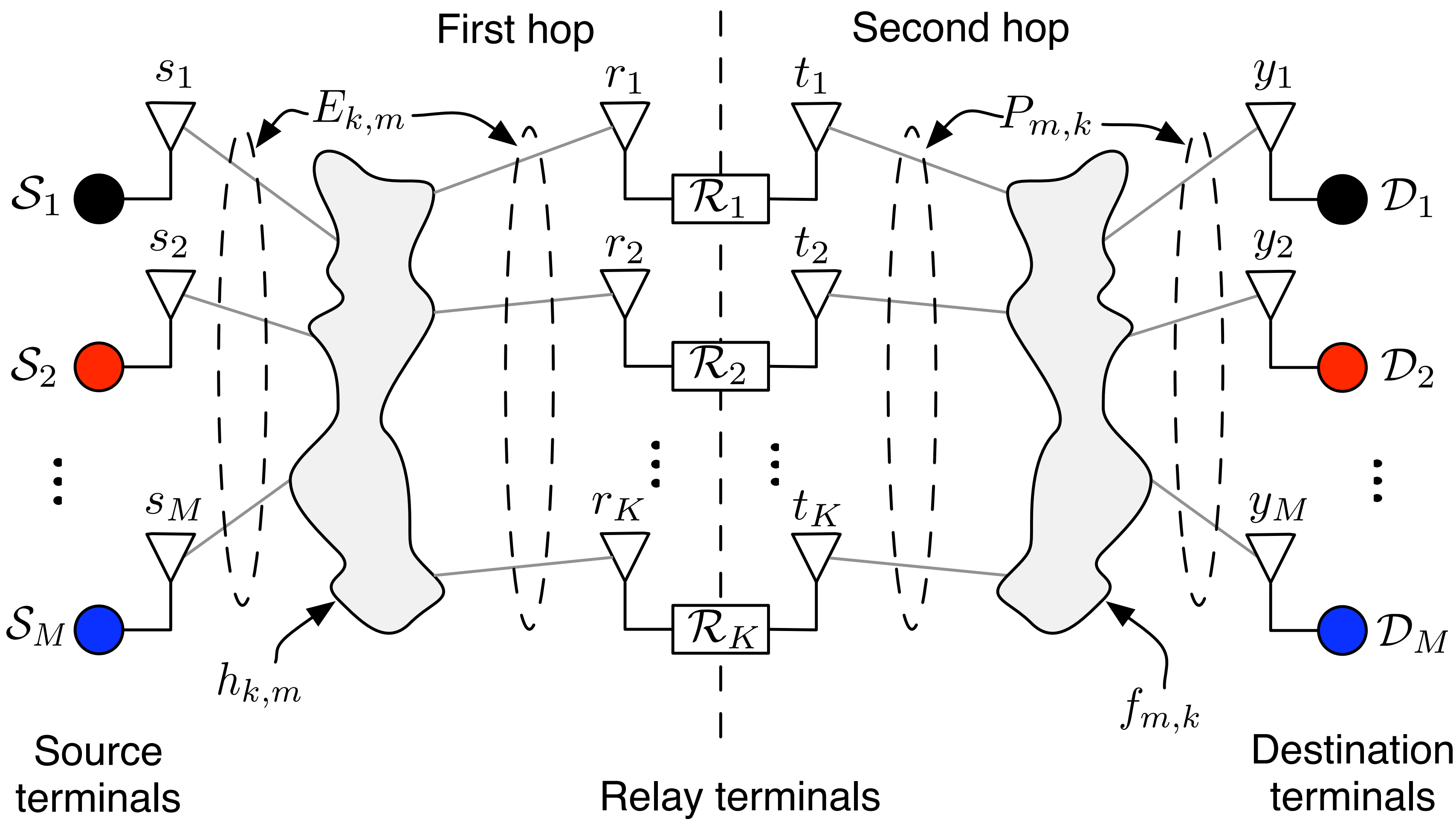
Motivation

Question: Can we achieve the cut set bound without cooperation?

Yes: [Bölcskei, Nabar 2004] show a protocol that

- For M **fixed** and $K \to \infty$ realizes **distributed orthogonalization**
- $C = (M/2)\log(K) + O(1)$

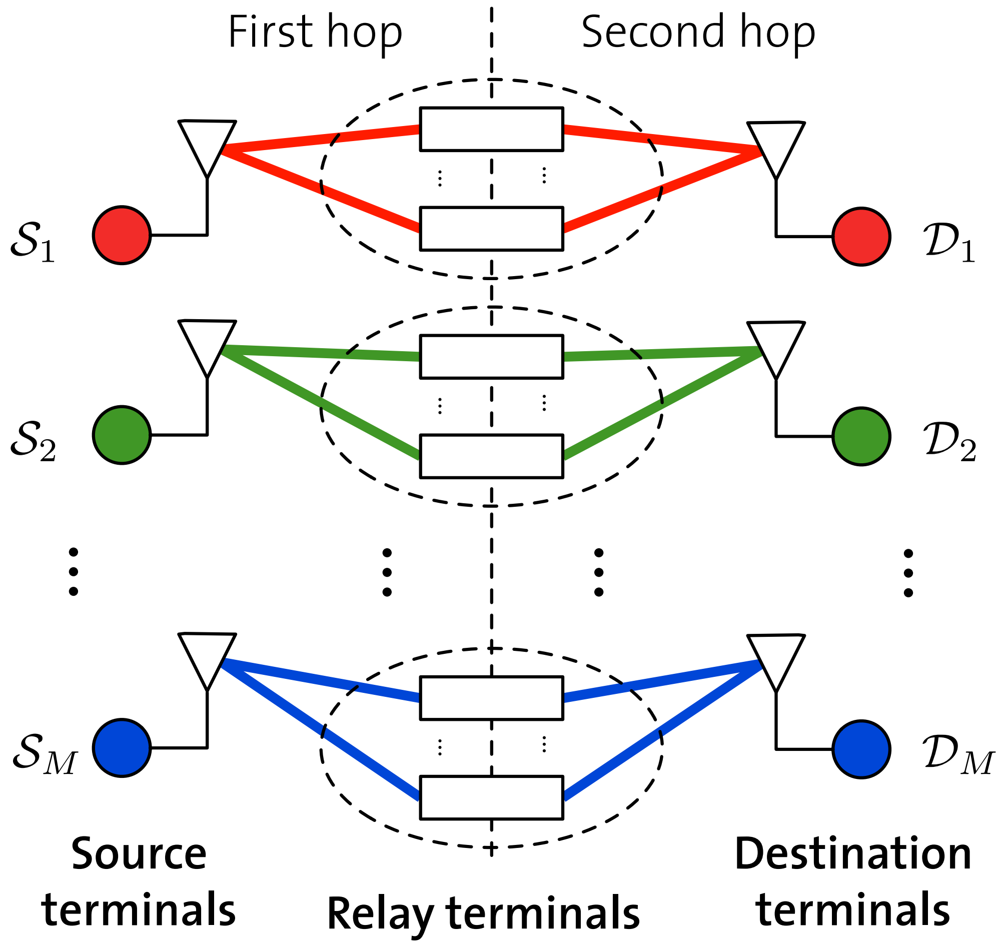
System and Channel Model



System and Channel Model Cont'd

- $r_k = \sum_{m=1}^{M} E_{k,m} h_{k,m} s_m + z_k$
- $y_m = \sum_{k=1}^{K} P_{m,k} f_{m,k} t_k + w_m$
- $\underline{E} \leq E_{k,m} \leq \overline{E}, \qquad \underline{P} \leq P_{m,k} \leq \overline{P} \qquad \forall k, m$
- $h_{k,m}, f_{m,k} \sim \mathcal{CN}(0, 1)$, i.i.d.
- $z_k, w_m \sim \mathcal{CN}(0, \sigma^2)$, i.i.d.
- Gaussian codebooks are used
- $\mathbb{E}[|s_m|^2] \leq 1/M, \;\; \mathbb{E}[|t_m|^2] \leq P_{\text{rel}}/K$

Protocol 1 (P1) from [Bölcskei, Nabar '04]



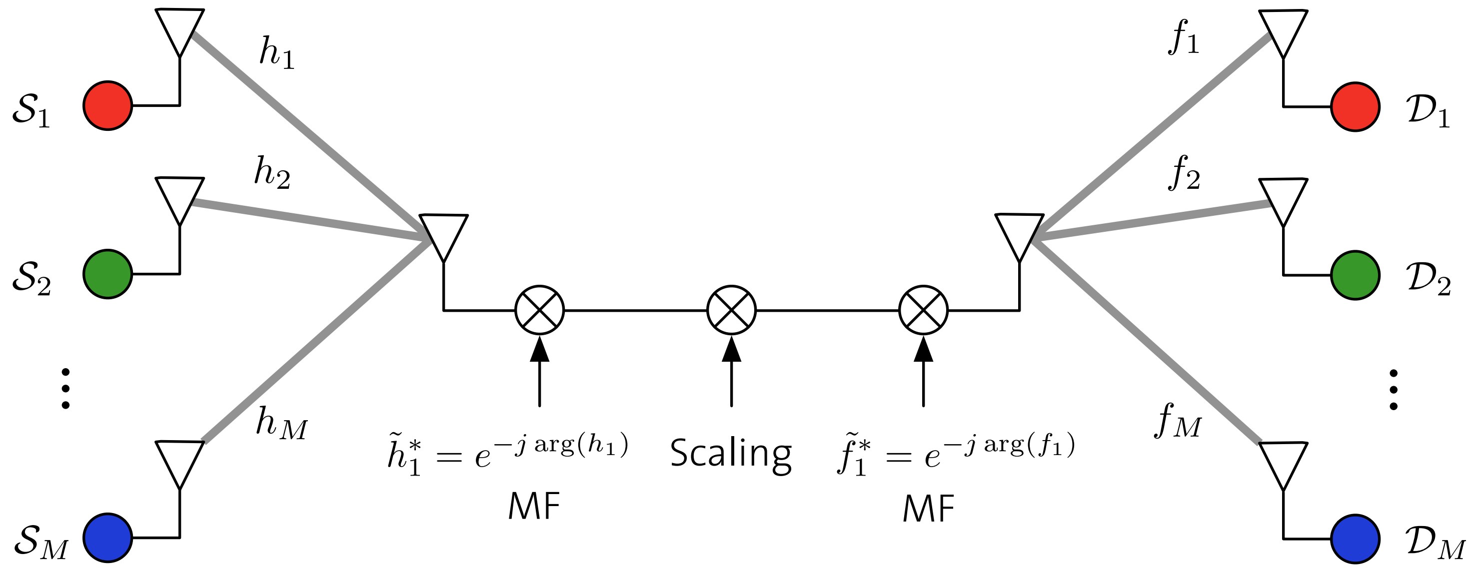
- K relay terminals are **partitioned** into M groups of equal size

  $\Rightarrow K/M$ relays in each group

- Each group is assigned to **one** S-D pair

- Each relay knows phases of its **assigned backward** and **forward** channels

Smart Scattering



Distributed multi-stream separation through **smart scatterers** performing **matched-filtering**

Capacity Scaling for large K and fixed M

- For fixed M and $K \to \infty$, lower bound approaches upper bound and the **network capacity** converges (w.p.1) to

$$C = \frac{M}{2}\log(K) + O(1)$$

- **Asymptotically** in K **cooperation between destination terminals** is **not needed to achieve network capacity**

Questions:

- Is **distributed orthogonalization** possible if **both** $M, K \to \infty$?
- If so, how K should scale with M?

[VM et al., 2005]: K should grow as M^3

For $M, K \to \infty$, the per S-D pair capacity scales as

$$C_{\mathrm{P1}} = \frac{1}{2}\log\left(1 + \Theta\left(\frac{K}{M^3}\right)\right)$$

Question: Is there protocol which requires **less relays** to realize **distributed orthogonalization**?

First…

Proof Techniques

I-O relation of $\mathcal{S}_m \to \mathcal{D}_m$ link

Independent decoding. I-O relation of $\mathcal{S}_m \to \mathcal{D}_m$ link is written as

$$y_m = \underbrace{s_m \sum_{k=1}^{K} a_k^{m,m}}_{\text{effective channel gain } (g_m)} + \underbrace{\sum_{\hat{m} \neq m} s_{\hat{m}} \sum_{k=1}^{K} a_k^{m,\hat{m}}}_{\text{interference } (i_m)} + \underbrace{\sum_{k=1}^{K} b_k^m z_k + w_m \sqrt{K}}_{\text{noise } (n_m)}$$

where

$$a_k^{m,\hat{m}} \sim \tilde{f}^*_{p(k),k} \, f_{m,k} \, \tilde{h}^*_{k,p(k)} \, h_{k,\hat{m}}$$

$$p(k) = m \quad \text{iff relay } k \text{ serves } m\text{th S-D pair}$$

Lower Bound

- I-O relation of $\mathcal{S}_m \to \mathcal{D}_m$ link

$$y_m = \mathbb{E}[g_m]s_m + \underbrace{(g_m - \mathbb{E}[g_m])}_{\tilde{g}_m} s_m + \underbrace{i_m + n_m}_{w_m}$$

  - effective channel gain has **non-zero mean**, i.e., $\mathbb{E}[g_m] > 0$
  - zero-mean w_m is not Gaussian
  - w_m and $\tilde{g}_m$ are not statistically independent

- Slight modification of a technique from [Médard, 2000] yields

$$I(y_m; s_m) \geq \log\left(1 + \frac{(\mathbb{E}[g_m])^2}{\mathbb{V}\mathrm{ar}[\tilde{g}_m] + \mathbb{V}\mathrm{ar}[w_m]}\right)$$

Outage Analysis: "Crystallization"

- Each destination terminal knows fading coefficients in entire network
- I-O relation of $\mathcal{S}_m \to \mathcal{D}_m$ link given by

$$y_m = g_m s_m + i_m + n_m$$

- Conditioned on $\{h_{k,m}, f_{m,k}\}_{\forall m,k}$ interference i_m and noise n_m are Gaussian

$$\Rightarrow \; I_m = \frac{1}{2}\log\left(1 + \mathsf{SINR}_m|_{\{h_{k,m},f_{m,k}\}}\right)$$

where

$$\mathsf{SINR}_m|_{\{h_{k,m},f_{m,k}\}} = \frac{|g_m|^2}{\sigma_i^2 + \sigma_n^2}$$

Goal: Analyze behavior of the random variable SINR when $M, K \to \infty$

Proof Techniques for Concentration Results

SINR of $\mathcal{S}_m \to \mathcal{D}_m$ link given by

$$\text{SINR} = \frac{\left|\sum_{k:p(k)=m} a_k^{m,m} + \sum_{k:p(k)\neq m} a_k^{m,m}\right|^2}{\sum_{\hat{m}\neq m}\left|\sum_{k=1}^{K} a_k^{m,\hat{m}}\right|^2 + \sigma^2 M \sum_{k=1}^{K} |b_k^m|^2 + KM\sigma^2}$$

- Consider each term in the numerator and denominator separately
- Use Chernoff bound to estimate large deviations from mean
  - gives **asymptotically tight** results
  - **independence** of summands is **required**, which is **not the case** for a's and b's

Main Tool: Truncation Lemma (thanks to O. Zeitouni)

Have to deal with sums of the form $S_N = \sum_{i=1}^{N} A_i X_i \phi_i$, where

- $\{X_i\}_{i=1}^{\infty}$ (not necessarily independent) with common cdf F_X
- i.i.d. $\{\phi_i\}_{i=1}^{\infty}$ $(-1 \leq \phi_i \leq 1)$
- positive and uniformly bounded coefficients $\{A_i\}_{i=1}^{\infty}$ $(0 \leq A_i \leq A^*)$
- for all $x \geq x_0 > 0$, we have $1 - F_X(x) + F_X(-x) \leq Ae^{-\alpha x^{\beta}}$

Then, for all N and t such that $\delta^2 \geq x_0$

$$\mathrm{P}\left\{|S_N - \mathbb{E}\{S_N\}| \geq \sqrt{N}\delta\right\} \leq 2\exp\left\{-\frac{2\delta^{2\beta/(\beta+2)}}{(A^*)^{2\gamma}}\right\} + NA\exp\left\{-\alpha\delta^{2\beta/(\beta+2)}\right\}$$

SINR is in Narrow Interval Around Mean with High Prob.

Theorem 1. *There exist constants* $C_1, C_2, C_3, C_4, C_5, C_6, M_0$ *and* K_0 *such that for any* $M \geq M_0$ *and* $K \geq K_0$ *for any* $x > 1$*, the probability* $\mathrm{P}_{\mathrm{out,P1}}$ *of the event* $\mathrm{SINR}_{\mathrm{P1}} \notin [L_{\mathrm{P1}}, U_{\mathrm{P1}}]$*, where*

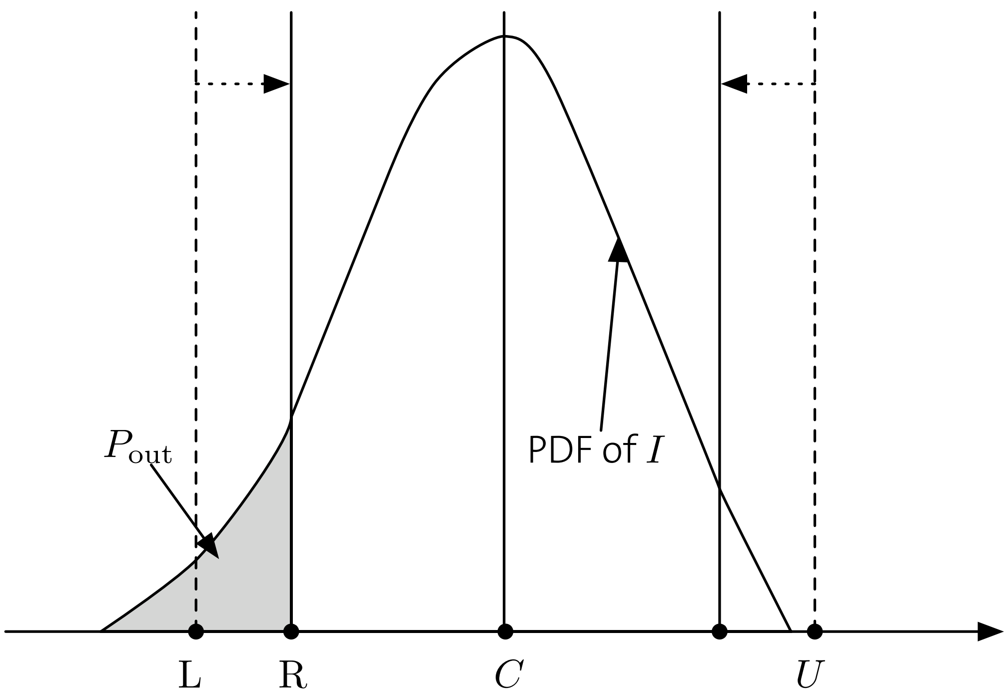
$$L_{\mathrm{P1}} = \frac{\pi^2}{16}\frac{\underline{P}\,\underline{E}^2}{\overline{P}\,\overline{E}^2}\frac{\left(\max\left[0, K - C_1 M\sqrt{K}x\right]\right)^2}{M^2(M-1)K + C_2 M^{5/2}Kx + C_3 M^3}$$

$$U_{\mathrm{P1}} = \frac{\pi^2}{16}\frac{\overline{P}\,\overline{E}^2}{\underline{P}\,\underline{E}^2}\frac{\left(K + C_4 M\sqrt{K}x\right)^2}{\max[0, M^2(M-1)K - C_5 M^{5/2}Kx] + C_6 M^3}$$

satisfies the following inequality

$$\mathrm{P}_{\mathrm{out,P1}} \leq \mathrm{Poly}_1(M,K)e^{-\Delta_1 x^{2/7}}$$

Outage Interpretation



- Fix $K = M^3$
- $C = \frac{1}{2}\log\left(1 + \frac{\pi^2}{16}\frac{\underline{P}\,\underline{E}^2}{\overline{P}\,\overline{E}^2}\right)$
- Choose $R < C$
- Choose P_{out} (gives x)
- Choose K such that $L = R$

Proof of Ergodic Capacity Upper Bound

- Upper bound on per S-D pair capacity

$$C \leq \frac{1}{2}\log(1 + \mathbb{E}\{\mathsf{SINR}(\mathbf{H}, \mathbf{F})\})$$

- Use that

$$\mathbb{E}\{X\} = \int_0^\infty x\, p_X(x) dx \leq \sum_{n=0}^{\infty} n\, \mathrm{P}\{X > n\}$$

- Using the tail behavior result for SINR, we get

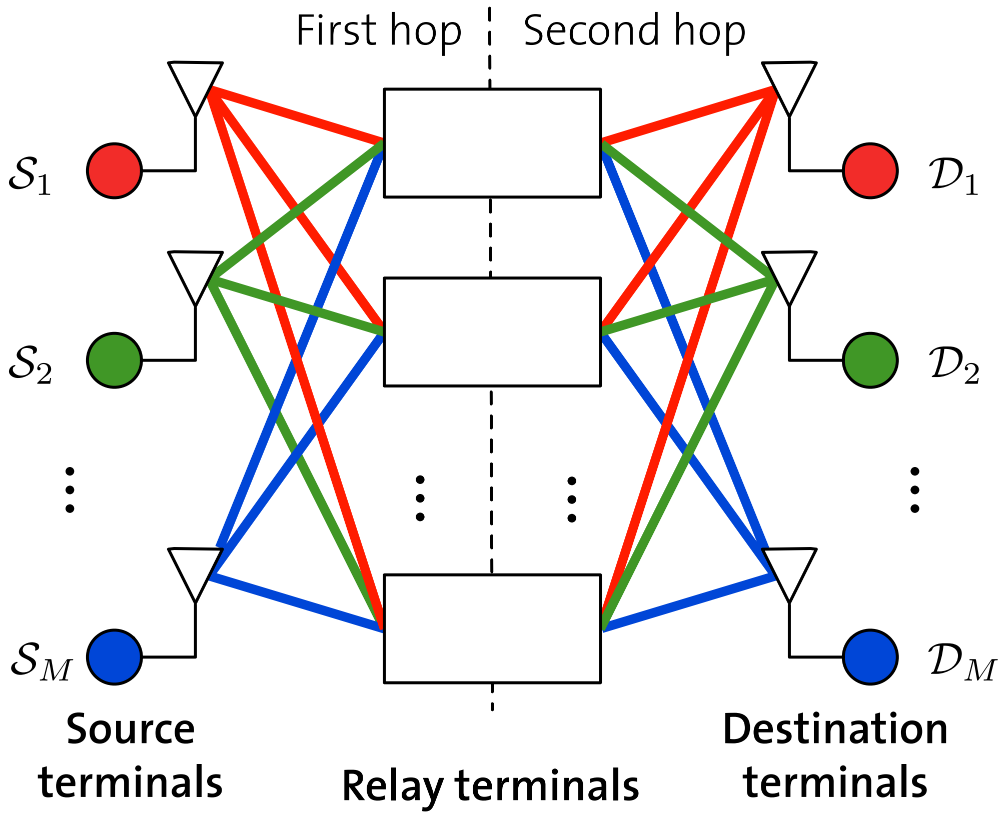
$$\mathrm{P}\left\{\mathsf{SINR} > \frac{(K + o(K)x)^2}{M^3 K - o(M^3 K)x}\right\} \leq \mathrm{Poly}(M, K)\, e^{-\Delta x^\beta}$$

Network "Crystallization"

- **SINRs** of effective channels $\mathcal{S}_m \to \mathcal{D}_m$ $(m = 1, 2, \ldots, M)$ **converge to deterministic limit** as $M, K \to \infty$
- **Per-stream diversity order** $\to \infty$ as $M, K \to \infty$
- **Individual SISO fading links** in the network **converge to independent AWGN links (network "crystallizes")**
- The **exponent** $2/7$, which characterizes the speed of convergence, is **unlikely to be fundamental**
- Theorems 1 can be reformulated to provide **bounds on outage probability**

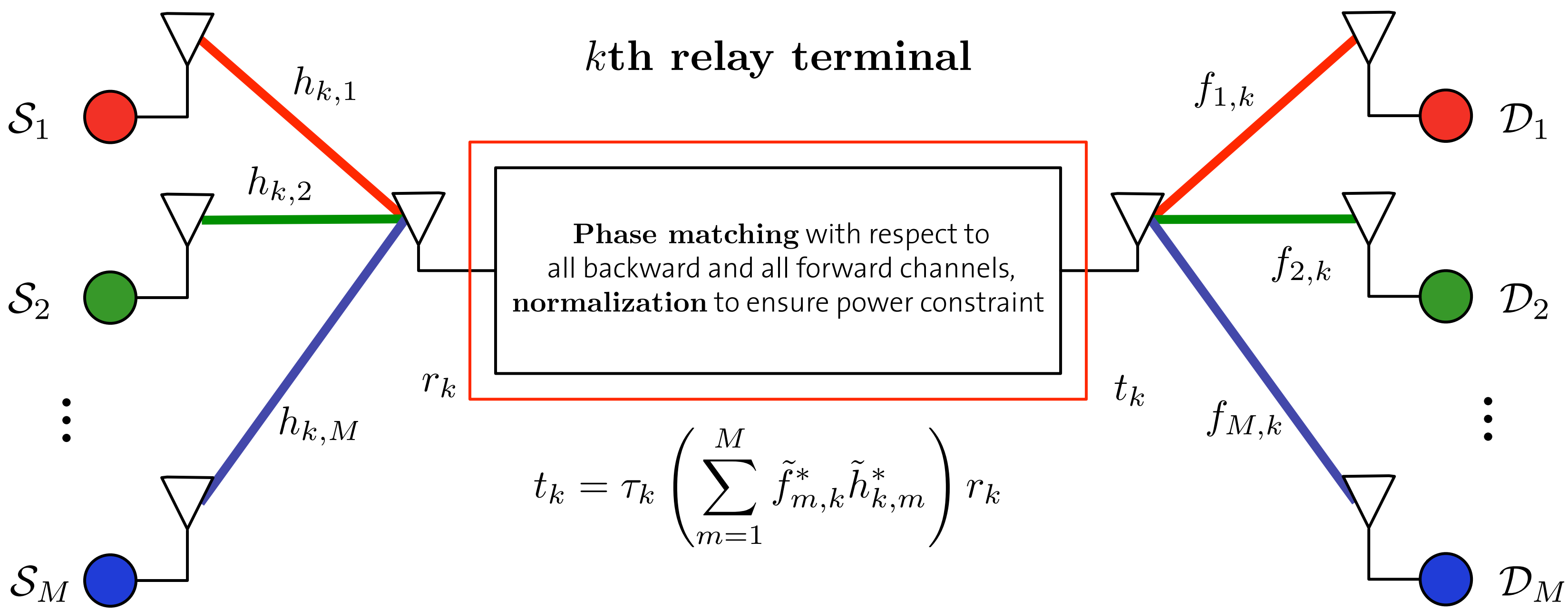
Can we do better than $K = M^3$?

Protocol 2 (P2) from [Dana and Hassibi, 2003]



- **No** relay **partitioning**
- Each relay knows phases of **all** M **backward** and **all** M **forward** channels

P2: Each Relay Assists All S-D Pairs



For $M, K \to \infty$, the per S-D pair capacity scales as

$$C_{\text{P2}} = \frac{1}{2} \log\left(1 + \Theta\left(\frac{K}{M^2}\right)\right)$$

Conclusions

- **Network decouples** if rate of growth of K as function of M is sufficiently fast

- P1 and P2 **trade amount of CSI at relays for required rate of growth of relays**

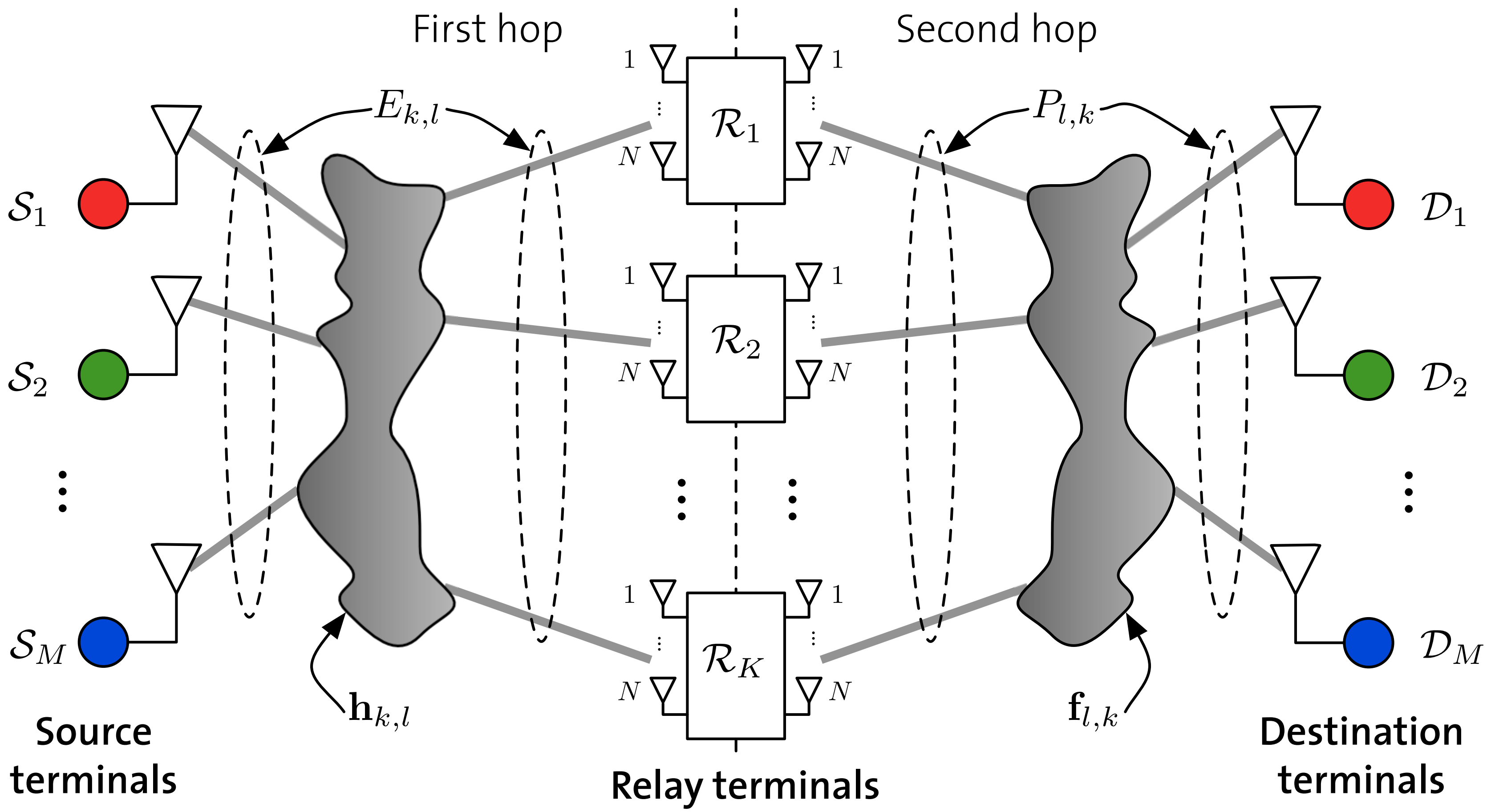
- The individual $\mathcal{S}_m \to \mathcal{D}_m$ **fading links converge to independent AWGN links** as $M, K \to \infty \Rightarrow$ **Network crystallizes**

- **Back from infinity**: Characterizing "**crystallization rate**" could serve as a **general tool** to study large wireless networks

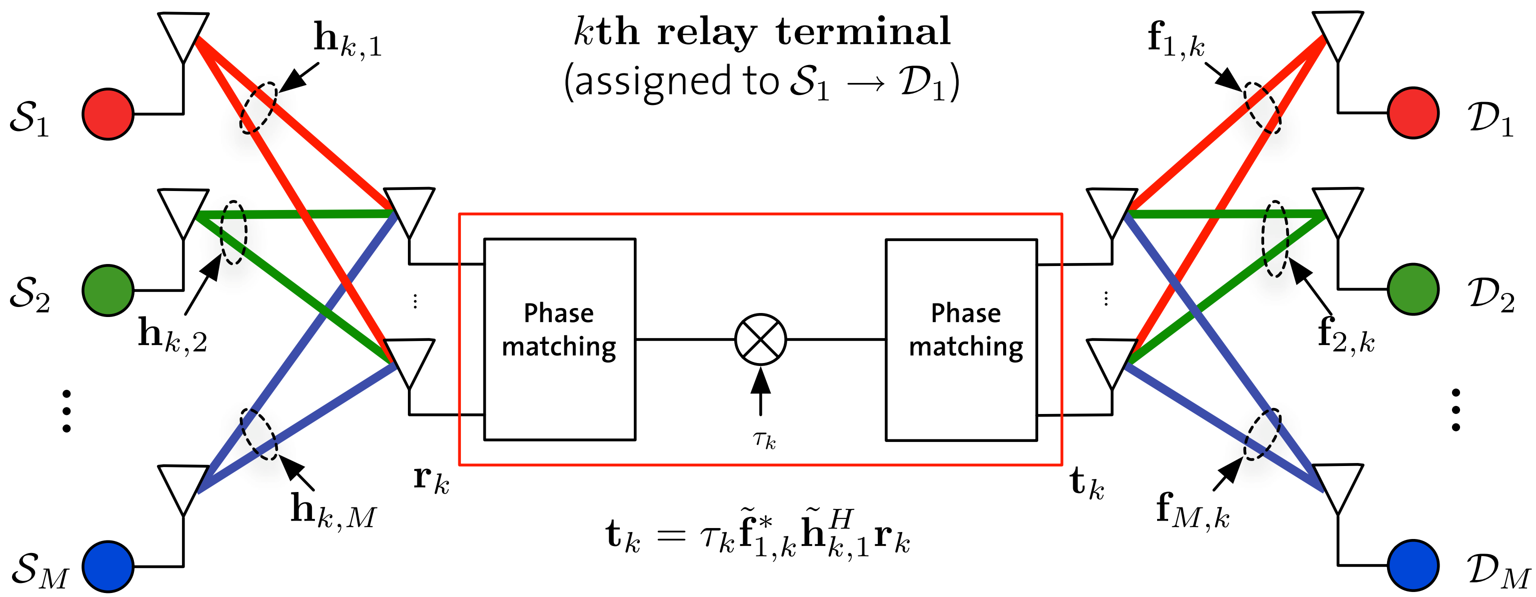
- Network capacity scaling for P2 is $\sqrt{T}$, where $T = 2M + K$

Cooperation

Interference Relay Network with Cooperation at Relays



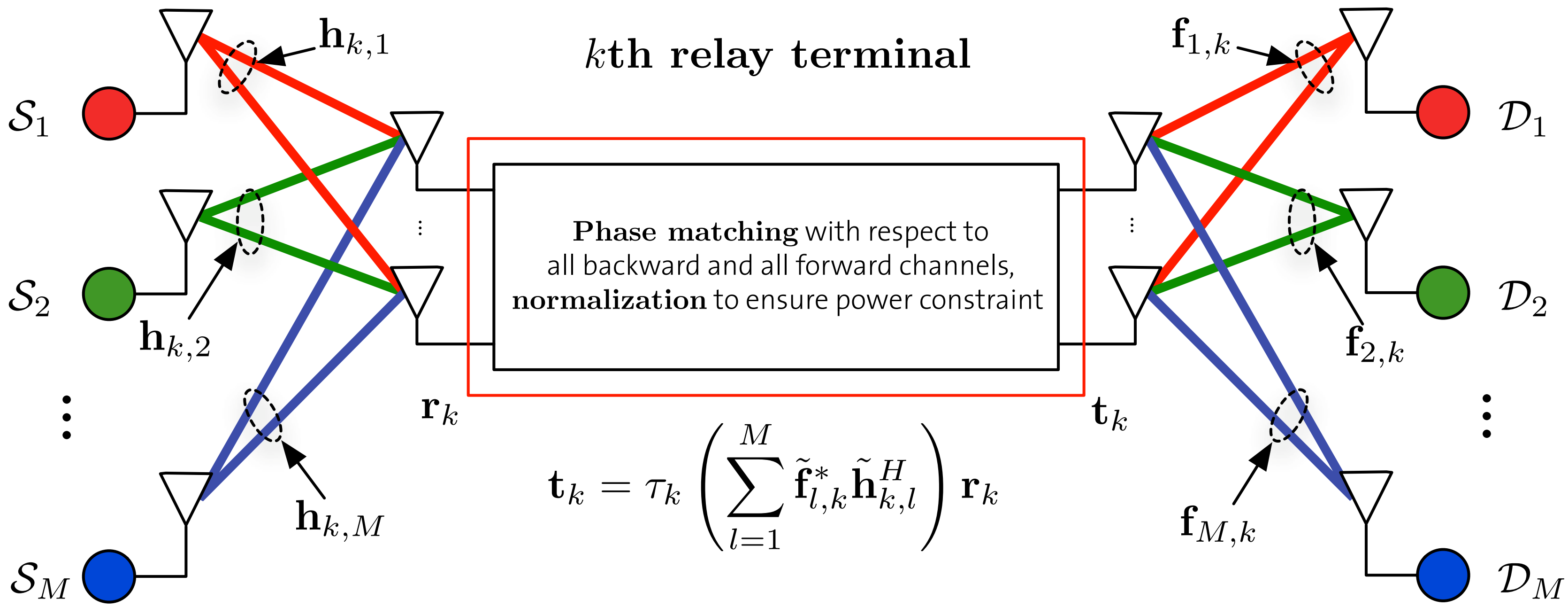
P1 with Cooperation



For $M, K \rightarrow \infty$, the per S-D pair capacity scales as

$$C_{\mathrm{P1}} = \frac{1}{2} \log\left(1 + \Theta\left(\frac{KN}{M^3}\right)\right)$$

P2 with Cooperation



For $M, K \to \infty$, the per S-D pair capacity scales as

$$C_{\text{P2}} = \frac{1}{2} \log \left(1 + \Theta \left(\frac{KN}{M^2} \right) \right)$$

Cooperation Increases Per-Stream Array Gain

- **Cooperation** at the relay level **increases** the **per-stream array gain**
- Per-stream array gain can be decomposed as $A = A_d A_c$, where
  - **Distributed array gain**

$$A_{d,\mathrm{P1}} = KN/M^3 \qquad A_{d,\mathrm{P2}} = KN/M^2$$

  - **Array gain due to cooperation** at relay level

$$A_{c,\mathrm{P1}} = A_{c,\mathrm{P2}} = N$$

Cooperation vs. No Cooperation

- Consider a network with a total of T relay antenna elements

- **No cooperation** at the relay level

$$C_{\mathrm{P1}}^{(nc)} = \frac{1}{2}\log\left(1 + \Theta\left(\frac{T}{M^3}\right)\right)$$

- **Cooperation** at the relay level in groups of N antenna elements

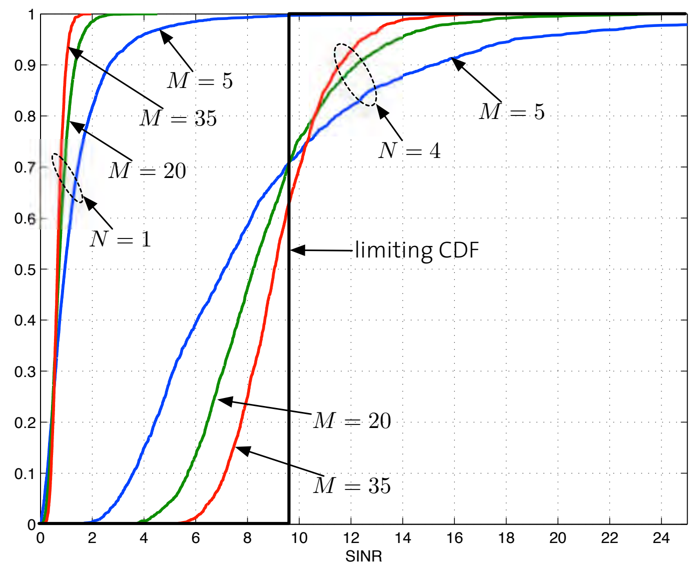
$$C_{\mathrm{P1}}^{(c)} = \frac{1}{2}\log\left(1 + \Theta\left(\frac{TN}{M^3}\right)\right)$$

Cooperation leads to N-fold **reduction** in **total number of relay antenna elements** needed to achieve given per S-D pair capacity

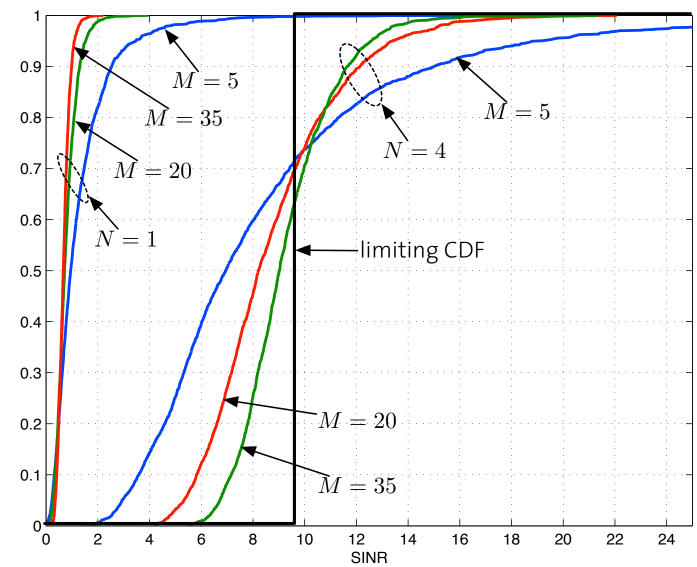
Thank You!

Backup

Convergence of SINR CDF to Step-Function



P1 for $K = M^3$



P2 for $K = M^2$