Capacity of Large Amplify and Forward Relay Networks

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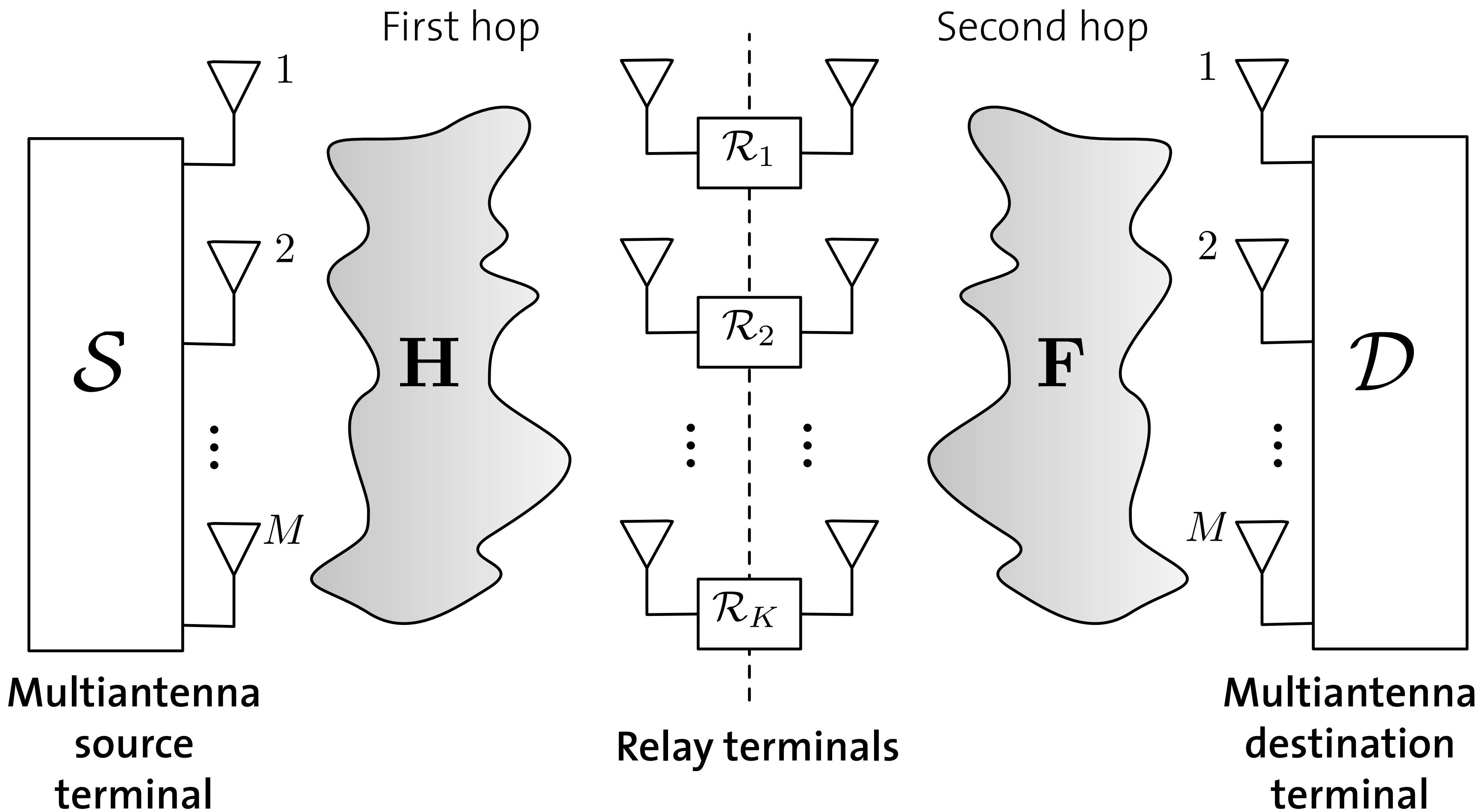
Amplify and Forward (AF) Relay Network

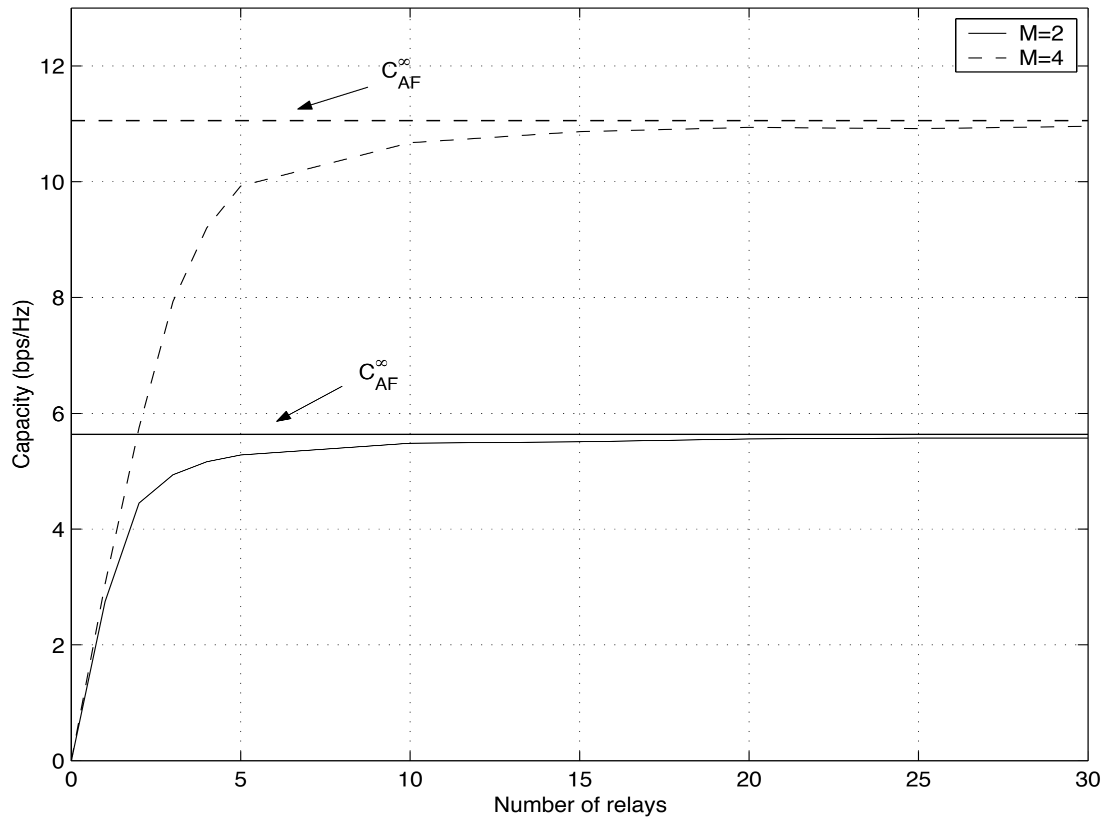


Large K Capacity of AF Relay Network for Finite M

- **Total power constraint** across source antennas and relay terminals
- **Receiver knows composite MIMO channel**
- For M fixed, in the limit $K \to \infty$, **AF relay network approaches point-to-point MIMO channel** with capacity [HB et. al., 2004]

$$C_{AF}^{\infty} = \frac{1}{2}\,\mathbb{E}_{\mathbf{H}}\big[\log\det\big(\mathbf{I} + \mathrm{SNR}\,\mathbf{H}_w\mathbf{H}_w^H\big)\big] = \frac{M}{2}\log(\mathsf{SNR}) + O(1)$$

Convergence of Capacity



Capacity vs. number of relays for the AF relay network

Generalization to $M \to \infty$

Assumptions

- Overall I-O relation: $\mathbf{y} = d\mathbf{F}\mathbf{H}\mathbf{s} + d\mathbf{F}\mathbf{n}_r + \mathbf{n}_d$
- Fixed receive SNR at each relay and at each destination node
- $\mathbf{H} \in \mathbb{C}^{K \times M}, \mathbf{F} \in \mathbb{C}^{M \times K}$
  - $\mathbf{H}$ … i.i.d. entries with mean 0 and variance $1/M$
  - $\mathbf{F}$ … i.i.d. entries with mean 0 and variance $1/K$
- $\mathbf{n}_r \in \mathbb{C}^{K \times 1}, \mathbf{n}_d \in \mathbb{C}^{M \times 1}$
  - $\mathbf{n}_r$ … i.i.d. $\mathcal{CN}(0, \sigma_n^2)$ noise at relays
  - $\mathbf{n}_d$ … i.i.d. $\mathcal{CN}(0, \sigma_n^2)$ noise at destination terminal
- Gaussian codebook, receiver knows $\mathbf{F}\mathbf{H}$ and $\mathbf{F}$

Capacity

- **Capacity** of the **effective MIMO channel** is given by

$$
\begin{aligned}
C &= \frac{1}{2}\,\mathbb{E}\left[\log\det\left(\mathbf{I} + \frac{d^2}{\sigma_n^2}\mathbf{H}^H\mathbf{F}^H\left(\mathbf{I} + d^2\mathbf{F}\mathbf{F}^H\right)^{-1}\mathbf{F}\mathbf{H}\right)\right] \\
&= \frac{1}{2}\,\mathbb{E}\left[\sum_{i=1}^{K}\log\left(1 + \frac{1}{\sigma_n^2}\lambda_i\right)\right]
\end{aligned}
$$

with

$$
\lambda_i = \lambda_i(\mathbf{H}\mathbf{H}^H\mathbf{T}) \qquad \text{and} \qquad \mathbf{T} = \mathbf{F}^H\left(\frac{1}{d^2}\mathbf{I} + \mathbf{F}\mathbf{F}^H\right)^{-1}\mathbf{F}
$$

- Need to study large M, K-behavior of $\lambda_i(\mathbf{H}\mathbf{H}^H\mathbf{T})$

Brief Review of Large Random Matrix Theory

- For an $M \times M$ random Hermitian matrix $\mathbf{X}$ define the *empirical eigenvalue distribution function (EEDF)* of $\mathbf{X}$ as

$$F_{\mathbf{X}}^{M}(x) = \frac{1}{M} \sum_{i=1}^{M} 1 \{\lambda_i(\mathbf{X}) \leq x\}$$

- From **Large Random Matrix Theory [Wigner, Silverstein, Bai, . . .]:**

  Under certain assumptions on $\mathbf{X}$, when $M \to \infty$, $F_{\mathbf{X}}^{M}(x)$ converges almost surely to a deterministic limit, i.e.,

$$F_{\mathbf{X}}^{M}(x) \xrightarrow{\text{a.s.}} F_{\mathbf{X}}(x)$$

  where $F_{\mathbf{X}}(x)$ is the *asymptotic EEDF*

Proof Program (for simplicity of exposition $K = M$)

Goal: Prove convergence of $F^M_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(x)$ and compute the corresponding asymptotic PDF $f_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(x)$

1. [Theorem (Silverstein, 1995)]: If $F^M_{\mathbf{T}}(x) \xrightarrow{\text{a.s.}} F_{\mathbf{T}}(x)$, then $F^M_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(x) \xrightarrow{\text{a.s.}} F_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(x)$ with the Stieltjes transform $m_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(z)$ given by the unique solution of a fixed-point equation (depends on $F_{\mathbf{T}}(x)$)

2. **Solve the fixed-point equation** and find $m_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(z)$

3. **Use the Stieltjes inversion formula** to compute $f_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(x)$

4. **Asymptotic per antenna capacity** given by

$$\frac{C}{M} = \frac{1}{2}\int_0^\infty \log\left(1 + \frac{1}{\sigma_n^2}x\right) f_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(x)\,dx$$

Computing $f_{\mathbf{T}}(x)$

- Singular value decomposition $\mathbf{F} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$

- $\mathbf{T}$ can be written as

$$\mathbf{T} = \mathbf{F}^H \left(\frac{1}{d^2}\mathbf{I} + \mathbf{F}\mathbf{F}^H \right)^{-1} \mathbf{F} = \mathbf{V} \operatorname{diag} \left\{ \frac{\lambda_i}{1/d^2 + \lambda_i} \right\}_{i=1}^{M} \mathbf{V}^H$$

- Marčenko ($\overset{\text{a.s.}}{\to}$ Marchenko)-Pastur law [Marčenko and Pastur, 1967] gives asymptotic PDF of eigenvalues of

$$\mathbf{F}^H\mathbf{F} = \mathbf{V} \operatorname{diag}\{\lambda_i\}_{i=1}^{M} \mathbf{V}^H$$

- Eigenvalues of $\mathbf{F}^H\mathbf{F}$ and $\mathbf{T}$ are related through a bijection $\Rightarrow$

$$f_{\mathbf{T}}(x) = \frac{1}{d^2(1-x)^2} f_{\mathbf{F}^H\mathbf{F}}\left(\frac{x}{d^2(1-x)} \right)$$

Computing $f_{\mathbf{T}}(x)$ (Cont'd)

Lemma 1. [Marčenko-Pastur] *If the matrix* $\mathbf{F} \in \mathbb{C}^{M,M}$ *has i.i.d. entries with mean* 0 *and variance* $1/M$*, then* $F^M_{\mathbf{F}^H\mathbf{F}}(x)$ *converges a.s., as* $M \to \infty$*, to a non-random* $F_{\mathbf{F}^H\mathbf{F}}(x)$ *with corresponding PDF*

$$f_{\mathbf{F}^H\mathbf{F}}(x) = \begin{cases} \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}, & 0 \le x \le 4 \\ 0, & \textit{otherwise} \end{cases}$$

Lemma 2. *Under the same conditions* $F_{\mathbf{T}}(x)$ *converges a.s., as* $M \to \infty$*, to a non-random* $F_{\mathbf{T}}(x)$ *with corresponding PDF*

$$f_{\mathbf{T}}(x) = \begin{cases} \frac{1}{2\pi d^2}\frac{1}{(1-x)^2}\sqrt{\frac{4d^2-(4d^2+1)x}{x}}, & 0 \le x \le 4d^2/(1+4d^2) \\ 0, & \textit{otherwise} \end{cases}$$

Brief Review of Stieltjes Transform

Let $F(x)$ be a distribution function

- **Stieltjes transform:**

$$m_F(z) := \int \frac{f(x)}{x-z}\,dx, \quad z \in \mathbb{C}^+ := \{z \in \mathbb{C} : \Im z > 0\}$$

- **Inversion formula:**

$$f(x) = \frac{1}{\pi} \lim_{y \to 0^+} \Im\left[m_F(x+iy)\right]$$

[Silverstein, 1995]

Assume that

- $\mathbf{H} \in \mathbb{C}^{M \times M}$ has i.i.d. elements with mean 0 and variance $1/M$
- $\mathbf{T} \in \mathbb{C}^{M \times M}$ is a random Hermitian nonnegative definite matrix, with $F_{\mathbf{T}}^{M}(x) \xrightarrow{\text{a.s.}} F_{\mathbf{T}}(x)$ on $[0, \infty)$ as $M \to \infty$
- $\mathbf{H}$ and $\mathbf{T}$ are independent

Then, $F_{\mathbf{H}\mathbf{H}^H\mathbf{T}}^{M}(x) \xrightarrow{\text{a.s.}} F_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(x)$, as $M \to \infty$, with Stieltjes transform satisfying

$$m_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(z) = -\int_{-\infty}^{\infty} \frac{f_{\mathbf{T}}(x)dx}{z\left(x\, m_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(z) + 1\right)}, \quad z \in \mathbb{C}^{+}$$

The solution of this equation is unique in $\mathbb{C}^{+}$

Putting the Pieces Together

- Putting the pieces together, we get

$$m_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(z) = -\frac{1}{2\pi d^2} \int_0^{\frac{4d^2}{(4d^2+1)}} \frac{\sqrt{4d^2-(4d^2+1)x}}{(1-x)^2\sqrt{x}} \frac{dx}{z\,(x\,m_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(z)+1)}$$

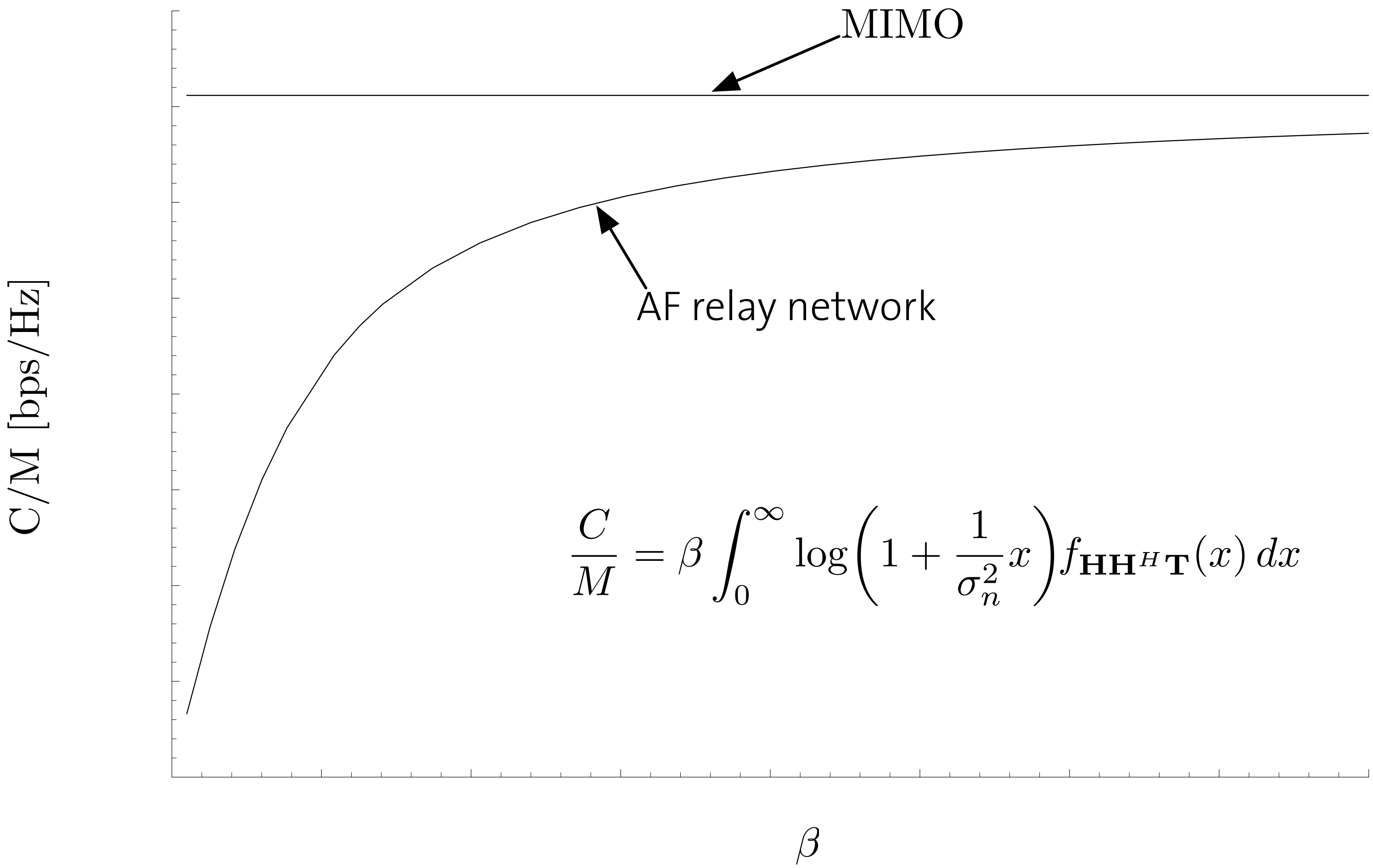
- $\Rightarrow m = m_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(z)$ satisfies the following equation of order 4:

$$d^2z^2m^4 + 2d^2z^2m^3 + (d^2z^2 + 2d^2z - z)m^2 + (2d^2z - 1)m + d^2 = 0$$

- Only one of the roots satisfies the initial equation

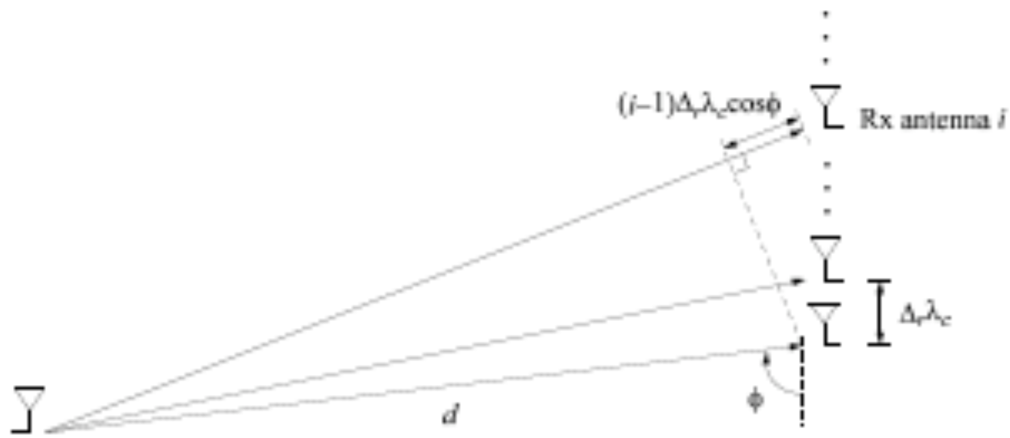
- Asymptotic PDF $f_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(x)$ can be computed **analytically** using the Stieltjes inversion formula

Asymptotic Capacity for $d = 1$ as Function of $\beta = K/M$



Motivation: Relays as active scatterers in poor scattering environment

Line of sight SIMO Flat channel



- λ_c ... wavelength
- $d_i \approx d + (i-1)\Delta_r \lambda_c \cos\phi$
- $\Omega = \cos\phi$

Channel gain is given by

$$h_i = a \exp\left(-\frac{j2\pi d_i}{\lambda_c}\right)$$

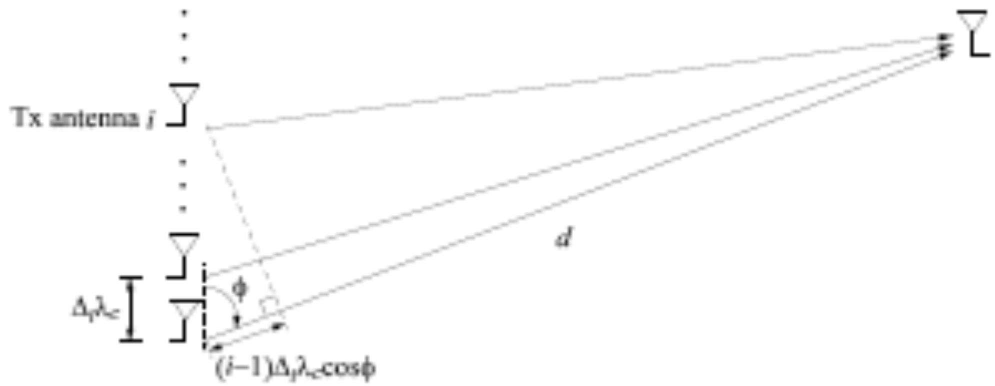
Line of sight SIMO Flat channel

Stacking channel gains in a vector $\mathbf{h} = [h_1, \ldots, h_{n_r}]^T$ we obtain

$$\mathbf{h} = a \exp\left(-\frac{j2\pi d}{\lambda_c}\right) \underbrace{\begin{bmatrix} 1 \\ \exp(-j2\pi\Delta_r\Omega) \\ \exp(-j2\pi 2\Delta_r\Omega) \\ . \\ . \\ \exp(-j2\pi(n_r-1)\Delta_r\Omega) \end{bmatrix}}_{\mathbf{e}_r(\Omega)}$$

and the channel is $\mathbf{y} = \mathbf{h}x + \mathbf{n}$

Line of sight MISO Flat channel



Stacking channel gains in a vector $\mathbf{h} = [h_1, \ldots, h_{n_r}]^T$ we obtain

$$\mathbf{h} = a\exp\left(-\frac{j2\pi d}{\lambda_c}\right)\underbrace{\begin{bmatrix} 1 \\ \exp(-j2\pi\Delta_t\Omega) \\ \exp(-j2\pi 2\Delta_t\Omega) \\ . \\ . \\ \exp(-j2\pi(n_t-1)\Delta_t\Omega) \end{bmatrix}}_{\mathbf{e}_t(\Omega)}, \qquad y = \mathbf{h}^*\mathbf{x} + n$$

Line of sight MIMO Flat channel is Rank 1

- The channel coefficients are

$$h_{ik} = a\exp(-j2\pi d_{ik}/\lambda_c)$$

- Distances can be approximated as

$$d_{ik} \approx d + (i-1)\Delta_r\lambda_c\cos\phi_r - (k-1)\Delta_t\lambda_c\cos_{\phi_t}$$

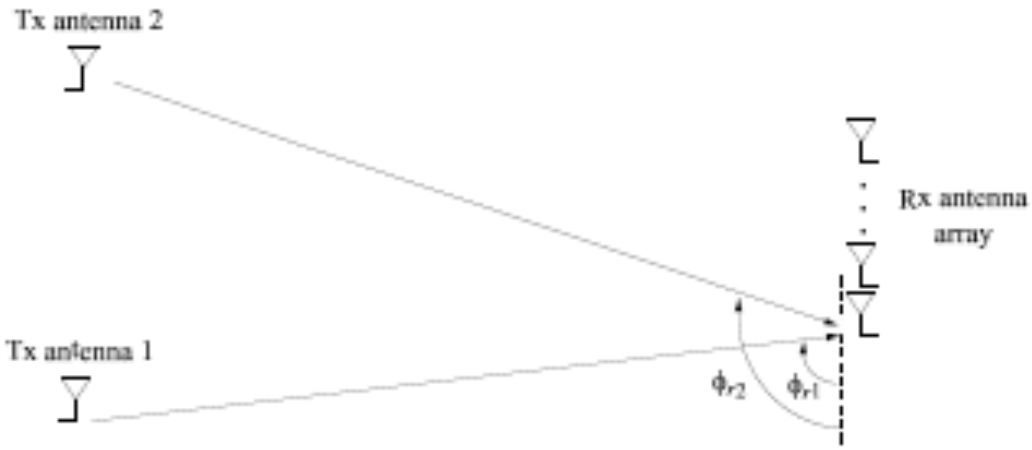
- $\Rightarrow$

$$h_{ik} = a\exp\left(-\frac{j\pi 2\pi d}{\lambda_c}\right)\exp(j2\pi(k-1)\Delta_t\Omega_t)\exp(j2\pi(i-1)\Delta_r\Omega_r)$$

- In matrix notation

$$\mathbf{H} = a\sqrt{n_t n_r}\exp\left(-\frac{j2\pi d}{\lambda_c}\right)\mathbf{e}_r(\Omega_r)\mathbf{e}_t(\Omega_t)^*$$

Geographically Separated antennas



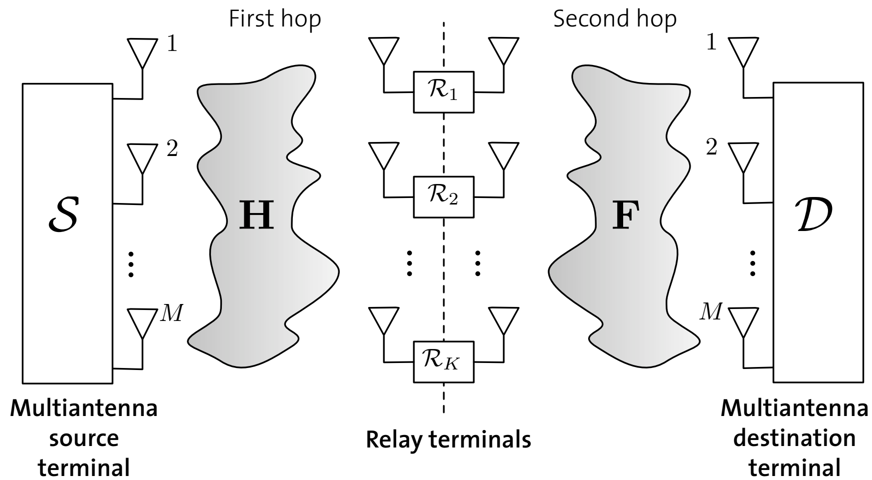
- Now the channel gain for each channel is

$$\mathbf{h}_k = a_k \sqrt{n_r} \exp\left(-\frac{j2\pi d_{1k}}{\lambda_c}\right) \mathbf{e}_r(\Omega_{rk}), \quad k = 1, 2$$

where $\Omega_{rk} = \cos\phi_{rk}$

- Channel matrix $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2]$ is full rank

Conclusion



- $\mathbf{HF}$ is full rank
- $\Rightarrow$ **Geographically separated relays** help to **improve poor scattering environment**