Mathematical methods for machine learning and signal processing SS 19

Problem set 2

Veniamin Morgenshtern, Helmut Bölcskei

Due date: 23. May 2019

Problem 1: Eigenvalue decomposition

1. Let $\mathbf{T}$ be an $N \times N$ Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_N$. What are the eigenvalues of the matrix $\mathbf{T}(\mathbf{I}+\mathbf{T})^{-1}$?

2. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are $N \times N$ Hermitian matrices with eigenvalues $\lambda_1^{\mathbf{A}}, \ldots, \lambda_N^{\mathbf{A}}$ and $\lambda_1^{\mathbf{B}}, \ldots, \lambda_N^{\mathbf{B}}$, respectively. Is it possible to say something about the eigenvalues of $\mathbf{AB}$?

3. Assume now that both matrices $\mathbf{A}$ and $\mathbf{B}$ have a common set of eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_N$. What can you say about the eigenvalues of $\mathbf{AB}$ now?

4. A circulant matrix is a matrix of the form

$$\mathbf{C} = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & \cdots & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}.$$

   (a) Find the eigenvectors and eigenvalues of a circulant matrix.

   (b) Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two $N \times N$ circulant matrices. Is it true that these matrices commute $\mathbf{C}_1\mathbf{C}_2 = \mathbf{C}_2\mathbf{C}_1$? If yes, give a proof; if no, find a counterexample.

Problem 2: Bandpass filter and orthogonal complement

Consider the space of $\mathcal{H}$-sequences with discrete-time Fourier transform (DTFT) supported on the interval $[f_1, f_2]$ with $-1/2 < f_1 < f_2 < 1/2$. Show that the orthogonal complement (in $\mathcal{H}$) of this space is the space of $\mathcal{H}$-sequences with discrete-time Fourier transform supported on the set $[-1/2, 1/2] \setminus [f_1, f_2]$.

Hint: Use the definition of the orthogonal complement and apply Parseval's theorem.

Problem 3: Tight frames

1. Prove that if $K \in \mathbb{Z} \setminus \{0\}$, then the set of vectors

$$\left\{ \mathbf{g}_k = \begin{bmatrix} 1 & e^{\mathrm{i}2\pi k/(KM)} & \cdots & e^{\mathrm{i}2\pi k(M-1)/(KM)} \end{bmatrix}^{\mathsf{T}} \right\}_{0 \le k < KM}$$

   is a tight frame for $\mathbb{C}^M$. Compute the frame bound.

2. Fix $T \in \mathbb{R} \setminus \{0\}$ and define for every $k \in \mathbb{Z}$

$$g_k(t) = \begin{cases} e^{\mathrm{i}2\pi kt/T}, & t \in [0, T] \\ 0, & \text{otherwise.} \end{cases}$$

Prove that $\{g_k(t)\}_{k\in\mathbb{Z}}$ is a tight frame for $\mathcal{L}^2([0, T])$, the space of square integrable functions supported on the interval $[0, T]$. Compute the frame bound.

Problem 4: Sampling theory I

Refer to Section 4 in Lecture notes on Frame Theory and consider the operator $\mathbb{A}$ defined as

$$\mathbb{A} : \{c_k\}_{k\in\mathbb{Z}} \rightarrow \sum_{k=-\infty}^{\infty} c_k h_{\mathrm{LP}}\left(\frac{t}{T} - k\right)$$

and the operator $\mathbb{T}$ defined as

$$\mathbb{T} : x \rightarrow \{\langle x, g_k\rangle\}_{k\in\mathbb{Z}},$$

1. Consider a sequence $\{a_k\}_{k\in\mathbb{Z}} \in \mathcal{R}(\mathbb{T})$ and show that $\mathbb{A}\{a_k\}_{k\in\mathbb{Z}} = \tilde{\mathbb{T}}^*\{a_k\}_{k\in\mathbb{Z}}$.
2. Consider a sequence $\{b_k\}_{k\in\mathbb{Z}} \in \mathcal{R}(\mathbb{T})^\perp$ and show that $\mathbb{A}\{b_k\}_{k\in\mathbb{Z}} = 0$.
3. Using the fact that every sequence $\{c_k\}_{k\in\mathbb{Z}}$ can be decomposed as $\{c_k\}_{k\in\mathbb{Z}} = \{a_k\}_{k\in\mathbb{Z}} + \{b_k\}_{k\in\mathbb{Z}}$ with $\{a_k\}_{k\in\mathbb{Z}} \in \mathcal{R}(\mathbb{T})$ and $\{b_k\}_{k\in\mathbb{Z}} \in \mathcal{R}(\mathbb{T})^\perp$, show that $\mathbb{A} = \tilde{\mathbb{T}}^*\mathbb{P}$ where $\mathbb{P} : \mathcal{H} \rightarrow \mathcal{H}$ is the orthogonal projection operator onto $\mathcal{R}(\mathbb{T})$.

Hints: Use the fact that $\mathcal{R}(\mathbb{T})$ is the space of $\mathcal{H}$-sequences with DTFT supported on the interval $[-BT, BT]$; use the characterization of $\mathcal{R}(\mathbb{T})^\perp$ developed in Problem 2; work in the DTFT domain.

Problem 5: Sampling theory II

Refer to Section 4 in Lecture notes on Frame Theory and use the ideas from the previous exercise to show that the operator $\mathbb{B}$ defined as

$$\mathbb{B} : \{c_k\}_{k\in\mathbb{Z}} \rightarrow \sum_{k=-\infty}^{\infty} c_k h_{\mathrm{out}}\left(\frac{t}{T} - k\right)$$

can be written as $\mathbb{B} = \mathbb{M}(\mathbb{I}_{l^2} - \mathbb{P})$, where $\mathbb{P} : l^2 \rightarrow l^2$ is the orthogonal projection operator onto $\mathcal{R}(\mathbb{T})$ with $\mathbb{T}$ defined as

$$\mathbb{T} : x \rightarrow \{\langle x, g_k\rangle\}_{k\in\mathbb{Z}},$$

and $\mathbb{M} : l^2 \rightarrow \mathcal{L}^2$ is the interpolation operator defined as

$$\mathbb{M} : \{c_k\}_{k\in\mathbb{Z}} \rightarrow \sum_{k=-\infty}^{\infty} c_k h_M\left(\frac{t}{T} - k\right).$$

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Problem 6: Weyl-Heisenberg (WH) frame in finite dimensions

Please read the short section 'Weyl-Heisenberg Frames' in the 'A short course on frame theory' before starting this exercise.

This is a computer exercise. The preferred language is Python. The point of the exercise is to understand what the abstract concept of the frame operator and the dual frame mean in linear algebra terms.

Consider the space $\mathbb{C}^M$. Take M to be a large number, such that your signals resemble continuous-time waveforms, but small enough such that your Matlab program works. Take a prototype vector $\mathbf{g} = [g[1] \cdots g[M]]^\mathsf{T} \in \mathbb{C}^M$. You can choose, for example, the $fir1(.)$ function, or discrete samples of the continuous-time Gaussian waveform $e^{-x^2/2}$. Next, fix the shift parameters $T, K \in \mathbb{N}$ in such a way that $L \triangleq M/T \in \mathbb{N}$. Now define

$$g_{k,l}[n] \triangleq g[(n - lT) \mod M]\, e^{\mathrm{i}2\pi kn/K}, \; k = 0, \dots, K-1, \; l = 0, \dots, L-1, \; n = 0, \dots, M-1$$

and construct a discrete-time WH set according to

$$\left\{ \mathbf{g}_{k,l} = [g_{k,l}[0] \;\; \cdots \;\; g_{k,l}[M-1]]^\mathsf{T} \right\}_{k=0,\dots,K-1,\, l=0,\dots,L-1}.$$

1. Show that the analysis operator $\mathbb{T} : \mathbb{C}^M \to \mathbb{C}^{KL}$ can be viewed as a $(KL \times M)$-dimensional matrix. Specify this matrix in terms of $\mathbf{g}, T, K$, and M.

2. Show that the adjoint of the analysis operator $\mathbb{T}^* : \mathbb{C}^{KL} \to \mathbb{C}^M$ can be viewed as an $(M \times KL)$-dimensional matrix. Specify this matrix in terms of $\mathbf{g}, T, K$, and M.

3. Specify the matrix corresponding to the frame operator $\mathbb{S}$ in terms of $\mathbf{g}, T, K$, and M. Call this matrix $\mathbf{S}$. Compute and store this $M \times M$ matrix in Matlab.

4. Given the matrix $\mathbf{S}$, check, if the WH system $\{\mathbf{g}_{k,l}\}_{k=0,\dots,K-1,\, l=0,\dots,L-1}$ you started from is a frame. Explain, how you can verify this.

5. Prove that for $K = M$ and $T = 1$ and for every prototype vector $\mathbf{g} \neq \mathbf{0}$, the set $\{\mathbf{g}_{k,l}\}_{k=0,\dots,K-1,\, l=0,\dots,L-1}$ is a frame for $\mathbb{C}^M$.

6. For the prototype vector $\mathbf{g}$ you have chosen, find two pairs of shift parameters (T_1, K_1) and (T_2, K_2) such that $\{\mathbf{g}_{k,l}\}_{k=0,\dots,K-1,\, l=0,\dots,L-1}$ is a frame for $T = T_1$ and $K = K_1$ and is not a frame for $T = T_2$ and $K = K_2$. For the case where $\{\mathbf{g}_{k,l}\}_{k=0,\dots,K-1,\, l=0,\dots,L-1}$ is a frame, compute the frame bounds.

7. Compute the dual prototype vector $\tilde{\mathbf{g}} = [\tilde{g}[1] \cdots \tilde{g}[M]]^\mathsf{T} = \mathbf{S}^{-1}\mathbf{g}$. Show that the dual frame $\{\tilde{\mathbf{g}}_{k,l}\}_{k=0,\dots,K-1,\, l=0,\dots,L-1}$ is given by time-frequency shifts of $\tilde{\mathbf{g}}$, i.e.,

$$\left\{ \tilde{\mathbf{g}}_{k,l} = [\tilde{g}_{k,l}[0] \;\; \cdots \;\; \tilde{g}_{k,l}[M-1]]^\mathsf{T} \right\}_{k=0,\dots,K-1,\, l=0,\dots,L-1}$$

with

$$\tilde{g}_{k,l}[n] \triangleq \tilde{g}[(n - lT) \mod M]\, e^{\mathrm{i}2\pi kn/K}, \; k = 0, \dots, K-1, \; l = 0, \dots, L-1, \; n = 0, \dots, M-1.$$

Problem 7: Frame expansion with noise

Let $\{\mathbf{g}_j\}_{j=1}^{M}$ be a tight frame for $\mathbb{C}^N$ $(N \leq M)$ with $\|\mathbf{g}_j\| = 1$ for all $1 \leq j \leq M$. The frame bound is $A = M/N$. We know that every $\mathbf{f} \in \mathbb{C}^N$ can be perfectly reconstructed from its frame expansion coefficients according to

$$\mathbf{f} = \frac{1}{A} \sum_{j=1}^{M} \langle \mathbf{f}, \mathbf{g}_j \rangle \, \mathbf{g}_j.$$

Now assume that the frame expansion coefficients are subject to noise:

$$\langle \mathbf{f}, \mathbf{g}_j \rangle \to \langle \mathbf{f}, \mathbf{g}_j \rangle + w_j$$

where $\{w_j\}_{j=1}^{M}$ are independent, zero-mean random variables with variance σ^2 each. After reconstruction we obtain in this case

$$\mathbf{f}_{\mathrm{w}} = \frac{1}{A} \sum_{j=1}^{M} (\langle \mathbf{f}, \mathbf{g}_j \rangle + w_j) \mathbf{g}_j.$$

Compute the mean square error (MSE) of the noisy reconstruction, defined as $\mathbb{E}\{\|\mathbf{f} - \mathbf{f}_{\mathrm{w}}\|^2\}$. How does the MSE depend on the redundancy $r = M/N$? Give an interpretation of the results.

Problem 8: Weyl-Heisenberg frame

Let $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be a smooth function such that its Fourier transform $\hat{\phi}$ satisfies the following properties:

1. Compact support: $\operatorname{supp} \hat{\phi} \subset [-1, 1]$
2. Partition-of-unity: $\sum_{k \in \mathbb{Z}} \left| \hat{\phi}(\nu - k) \right|^2 = 1$ for all $\nu \in \mathbb{R}$.

Show that the Weyl-Heisenberg system defined as $\{\mathbb{T}_{l/2} \mathbb{M}_k\}_{k,l \in \mathbb{Z}}$, with $\Phi = (1/\sqrt{2})\phi$, forms a tight frame with frame bound $A = 1$. Here, $\mathbb{T}_{k/2}$ and $\mathbb{M}_l$ denote the translation and modulation operators, respectively, i.e.,

$$\mathbb{T}_{k/2} \Phi(t) = \Phi(t - k/2)$$
$$\mathbb{M}_l \Phi(t) = e^{2i\pi l t} \Phi(t).$$

Hint: Observe that $\widehat{\mathbb{T}_{l/2} \mathbb{M}_k} \Phi = \mathbb{M}_{-l/2} \mathbb{T}_k$ and use Parseval's equality. You might use the fact that $\{e_{l/2}/\sqrt{2}\}_{l \in \mathbb{Z}}$, where $e_l(t) = e^{2i\pi l t}$ for all $t \in \mathbb{R}$, is an orthonormal basis for $L^2[k-1, k+1]$ for all $k \in \mathbb{Z}$.

Problem 9: Wavelet frame

Let $\phi, \psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be functions whose Fourier transforms satisfy the following properties:

1. $\operatorname{supp} \hat{\phi} \subset [-1, 1]$ and $\operatorname{supp} \hat{\psi} \subset [-2, -1/2] \cup [1/2, 2]$
2. $\left| \hat{\phi}(\nu) \right|^2 + \sum_{j=0}^{\infty} \left| \hat{\psi}(2^{-j} \nu) \right|^2 = 1$ for all $\nu \in \mathbb{R}$.

Define for $j \in \mathbb{N}$ and $k \in \mathbb{Z}$

$$\forall t \in \mathbb{R},\ \psi_{-1,k}(t) = \phi(t - k/2)/\sqrt{2}$$
$$\forall t \in \mathbb{R},\ \psi_{j,k}(t) = 2^{j/2-1}\psi(2^j t - k/4).$$

Show that $\psi_{j\geq -1,k\in\mathbb{Z}}$ forms a tight frame for $L^2(\mathbb{R})$ with frame bound 1.

Hint: see the solution to the previous exercise.