Mathematical methods for machine learning and signal processing SS 19

Solutions to problem set 1

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Problem 1: Overcomplete expansion in $\mathbb{R}^2$

1. Consider the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{e}_3 = \mathbf{e}_1 - \mathbf{e}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Our goal is to find vectors $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$ such that

$$\mathbf{x} = \langle \mathbf{x}, \tilde{\mathbf{e}}_1 \rangle \, \mathbf{e}_1 + \langle \mathbf{x}, \tilde{\mathbf{e}}_2 \rangle \, \mathbf{e}_2 + \langle \mathbf{x}, \tilde{\mathbf{e}}_3 \rangle \, \mathbf{e}_3 = \underbrace{\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \tilde{\mathbf{e}}_1^\mathsf{T} \\ \tilde{\mathbf{e}}_2^\mathsf{T} \\ \tilde{\mathbf{e}}_3^\mathsf{T} \end{bmatrix} \mathbf{x}.$$

In order to find these vectors, we are looking for a right inverse of the matrix $\mathbf{A}$. One possible right inverse can be found by noting that

$$\mathbf{A} \underbrace{\mathbf{A}^\mathsf{T} (\mathbf{A}\mathbf{A}^\mathsf{T})^{-1}}_{\text{right inverse}} = \mathbf{I}.$$

First we calculate

$$\mathbf{A}\mathbf{A}^\mathsf{T} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

and the inverse

$$(\mathbf{A}\mathbf{A}^\mathsf{T})^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

and finally

$$\begin{bmatrix} \tilde{\mathbf{e}}_1'^\mathsf{T} \\ \tilde{\mathbf{e}}_2'^\mathsf{T} \\ \tilde{\mathbf{e}}_3'^\mathsf{T} \end{bmatrix} = \mathbf{A}^\mathsf{T} (\mathbf{A}\mathbf{A}^\mathsf{T})^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}.$$

The vectors $\tilde{\mathbf{e}}_1', \tilde{\mathbf{e}}_2', \tilde{\mathbf{e}}_3'$ are given by

$$\tilde{\mathbf{e}}_1' = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}, \tilde{\mathbf{e}}_2' = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}, \tilde{\mathbf{e}}_3' = \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}.$$

Comparing to the given set of vectors $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$ we find

$$\tilde{\mathbf{e}}_1 = 2\mathbf{e}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{e}}_2 = -\mathbf{e}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \tilde{\mathbf{e}}_3 = -\mathbf{e}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

It should be emphasized that the right inverse is not unique: the system of equations

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}}_{\mathbf{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has infinitely many solutions of the form

$$\mathbf{B} = \begin{bmatrix} 1-\lambda & -\gamma \\ \lambda & 1+\gamma \\ \lambda & \gamma \end{bmatrix}$$

for any $\lambda, \gamma \in \mathbb{R}$. Any such matrix $\mathbf{B}$ is a valid right inverse of $\mathbf{A}$, which generates in general different set of vectors $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$.

2. Assume that $\mathbf{x}$ can be written in the form

$$\mathbf{x} = \left\langle \mathbf{x}, \mathbf{e}_1' \right\rangle \tilde{\mathbf{e}}_1 + \left\langle \mathbf{x}, \mathbf{e}_2' \right\rangle \tilde{\mathbf{e}}_2 = \begin{bmatrix} \tilde{\mathbf{e}}_1 & \tilde{\mathbf{e}}_2 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1'^{\mathsf{T}} \\ \mathbf{e}_2'^{\mathsf{T}} \end{bmatrix} \mathbf{x} = \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & \sqrt{2} \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \mathbf{e}_1'^{\mathsf{T}} \\ \mathbf{e}_2'^{\mathsf{T}} \end{bmatrix} \mathbf{x}.$$

$\mathbf{A}$ is a square, non-singular matrix and has, therefore, a unique inverse

$$\mathbf{A}^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{e}}_1^{\mathsf{T}} \\ \tilde{\mathbf{e}}_2^{\mathsf{T}} \end{bmatrix} \Rightarrow \mathbf{e}_1' = \mathbf{e}_1,\ \mathbf{e}_2' = \mathbf{e}_2.$$

We conclude that $\mathbf{x}$ can be represented in the form

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{e}_1 \rangle \tilde{\mathbf{e}}_1 + \langle \mathbf{x}, \mathbf{e}_2 \rangle \tilde{\mathbf{e}}_2$$

and this expansion is unique.

Problem 2: Equality in the Cauchy-Schwarz inequality

First, here is a proof of Cauchy-Schwarz inequality. Assume that $g \neq \mathbf{0}$, otherwise the inequality is trivially true. Define $\lambda = \langle f, g \rangle / \|g\|^2$ Then:

$$\begin{aligned} 0 &\leq \|f - \lambda g\|^2 \\ &= \langle f, f \rangle - \langle \lambda g, f \rangle - \langle f, \lambda g \rangle + \langle \lambda g, \lambda g \rangle \\ &= \|f\|^2 - \lambda \langle g, f \rangle - \lambda^* \langle f, g \rangle + \lambda\lambda^* \langle g, g \rangle \\ &= \|f\|^2 - \lambda \langle f, g \rangle^* - \lambda^* \langle f, g \rangle + \lambda\lambda^* \|g\|^2 \\ &= \|f\|^2 - \frac{|\langle f, g \rangle|^2}{\|g\|^2} - \frac{|\langle f, g \rangle|^2}{\|g\|^2} + \frac{|\langle f, g \rangle|^2}{\|g\|^2} \\ &= \|f\|^2 - \frac{|\langle f, g \rangle|^2}{\|g\|^2}. \end{aligned}$$

Therefore, $\left|\langle f, g \rangle\right| \leq \|f\| \|g\|$.

Next, let's solve the problem. Let's assume $\|f\| = \|g\| = 1$ and $\langle f, g\rangle = 1$, then from

$$\langle f - g, g\rangle = \langle f, g\rangle - \|g\| = 0$$

we conclude that $f - g$ is orthogonal to g. Applying Pythagoras theorem to $f = f - g + g$ we can write

$$\begin{aligned}
&\|f\|^2 = \|f - g\|^2 + \|g\|^2 \\
&\Rightarrow \|f - g\|^2 = 0 \\
&\Rightarrow f = g.
\end{aligned}$$

In the general case $\|f\| \neq 1$ and $\|g\| \neq 1$, we can do the following

$$\begin{aligned}
&|\langle f, g\rangle| = \|f\|\|g\| \\
&\Rightarrow \left|\left\langle \frac{f}{\|f\|}, \frac{g}{\|g\|}\right\rangle\right| = 1
\end{aligned}$$

We can multiply f by $e^{\mathrm{i}\phi}$ such that
$$\left\langle \frac{e^{\mathrm{i}\phi} f}{\|f\|}, \frac{g}{\|g\|}\right\rangle = 1$$
define $\tilde{f} = e^{\mathrm{i}\phi}\frac{f}{\|f\|}$ and $\tilde{g} = \frac{g}{\|g\|}$. Now we have the first case again

$$\begin{aligned}
&\left\langle \tilde{f}, \tilde{g}\right\rangle = 1 \\
&\|\tilde{f}\| = \|\tilde{g}\| = 1 \\
&\Rightarrow \tilde{f} = \tilde{g} \\
&\Rightarrow f = e^{-\mathrm{i}\phi}\frac{\|f\|}{\|g\|} g = cg.
\end{aligned}$$

Problem 3: Useful identities in a Hilbert Space

1. Parallelogram law

$$\begin{aligned}
\|f + g\|^2 + \|f - g\|^2 &= \langle f + g, f + g\rangle + \langle f - g, f - g\rangle \\
&= \langle f, f + g\rangle + \langle g, f + g\rangle + \langle f, f - g\rangle - \langle g, f - g\rangle \\
&= \langle f, f\rangle + \langle f, g\rangle + \langle g, f\rangle + \langle g, g\rangle + \langle f, f\rangle - \langle f, g\rangle - \langle g, f\rangle + \langle g, g\rangle \\
&= 2\|f\|^2 + 2\|g\|^2 + \langle f, g\rangle + \langle g, f\rangle^* - \langle f, g\rangle - \langle f, g\rangle^* \\
&= 2\|f\|^2 + 2\|g\|^2 + 2\Re\langle f, g\rangle - 2\Re\langle f, g\rangle \\
&= 2\left(\|f\|^2 + \|g\|^2\right)
\end{aligned}$$

2. Polarization identity

$$
\begin{aligned}
&\frac{1}{4}\left(\|f+g\|^2-\|f-g\|^2+\mathrm{i}\left[\|\frac{f}{\mathrm{i}}+g\|^2-\|\frac{f}{\mathrm{i}}-g\|^2\right]\right)\\
=&\frac{1}{4}\left(\langle f+g,f+g\rangle-\langle f-g,f-g\rangle+i\left[\left\langle\frac{f}{i}+g,\frac{f}{i}+g\right\rangle-\left\langle\frac{f}{i}-g,\frac{f}{i}-g\right\rangle\right]\right)\\
=&\frac{1}{4}(\|f\|^2+\|g\|^2+2\Re\langle f,g\rangle-\|f\|^2-\|g\|^2+2\Re\langle f,g\rangle\\
&+i\left[-\|f\|^2+\|g\|^2-i\langle f,g\rangle+i\langle g,f\rangle+\|f\|^2-\|g\|^2-i\langle f,g\rangle+i\langle g,f\rangle\right])\\
=&\frac{1}{4}\left(4\Re\langle f,g\rangle+4\mathrm{i}\Im\langle f,g\rangle\right)=\langle f,g\rangle
\end{aligned}
$$

Problem 4: Discrete Fourier Transform (DFT) as a signal expansion

Define the basis functions

$$e_k[n]=\frac{1}{\sqrt{N}}e^{i2\pi\frac{k}{N}n},\quad k=0,1,\ldots,N-1.$$

They form an ONS as shown by

$$
\begin{aligned}
\langle e_k,e_l\rangle&=\sum_{n=0}^{N-1}e_k[n]e_l[n]^*=\sum_{n=0}^{N-1}\frac{1}{\sqrt{N}}e^{i2\pi\frac{k}{N}n}\frac{1}{\sqrt{N}}e^{-i2\pi\frac{l}{N}n}=\frac{1}{N}\sum_{k=0}^{N-1}e^{i\frac{2\pi}{N}(k-l)n}\\
&=\begin{cases}\frac{1}{N}\frac{e^{i2\pi(k-l)}-1}{e^{i(k-l)\frac{2\pi}{N}}-1}=0, & k\neq l\\ \frac{1}{N}\sum_{k=0}^{N-1}1=1, & k=l.\end{cases}
\end{aligned}
$$

We have N functions in $\mathbb{C}^N$ that form an ONS, thus, they form an ONB. Therefore every signal can be expressed as

$$f[n]=\sum_{k=0}^{N-1}\langle f,e_k\rangle\, e_k[n]$$

where

$$\langle f,e_k\rangle=\sum_{n=0}^{N-1}f[n]e_k[n]^*=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}f[n]e^{-i2\pi\frac{k}{N}n}=\widehat{f}[n].$$

Therefore we see that the inverse of the DFT is given by

$$f[n]=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\widehat{f}[n]e^{i2\pi\frac{k}{N}n}.$$

From the lecture we know that an ONB is a tight and exact frame with frame bounds $A=B=1$.

Problem 5: Unitary transformation of a frame

Condition that $\{g_j\}_{j\in\mathcal{J}}$ is a frame means that there exist $A>0$ and $B<\infty$ such that for any $f\in\mathcal{H}$

$$A\|f\|^2\leq\sum_j|\langle f,g_j\rangle|^2\leq B\|f\|^2.$$

Then we have

$$\sum_j |\langle f, \mathbb{U} g_j \rangle|^2 = \sum_j |\langle \mathbb{U}^* f, g_j \rangle|^2 \le B \|\mathbb{U}^* f\|^2$$
$$\le B\|\mathbb{U}^*\|^2 \|f\|^2$$
$$= B\|f\|^2,$$

which establishes the upper frame bound. Next,

$$A\|f\|^2 = A\left\langle \mathbb{I} f, f\right\rangle = A\left\langle \mathbb{U}\mathbb{U}^* f, f\right\rangle = A\left\langle \mathbb{U}^* f, \mathbb{U}^* f\right\rangle = A\|\mathbb{U}^* f\|^2 \le \sum_j |\langle \mathbb{U}^* f, g_j\rangle|^2 = \sum_j |\langle f, \mathbb{U} g_j\rangle|^2,$$

which establishes the lower frame bound. Therefore, $\{\mathbb{U} g_j\}_{j\in\mathcal{J}}$ is a frame for $\mathcal{H}$ with the same frame bounds A and B. We have used in the proof the properties of a unitary operator $\mathbb{U}$: $\mathbb{U}\mathbb{U}^* = \mathbb{U}^*\mathbb{U} = \mathbb{I}$ and $\|\mathbb{U}\| = \|\mathbb{U}^*\| = 1$.

Problem 6: Redundancy of a frame

(a) We have that for every $\mathbf{f} \in \mathbb{C}^M$

$$A\|\mathbf{f}\|^2 = \sum_{j=1}^{N} |\langle \mathbf{f}, \mathbf{g}_j \rangle|^2. \tag{1}$$

Now let $\{\mathbf{e}_1, \ldots, \mathbf{e}_M\}$ be an ONB for $\mathbb{C}^M$ and expand $\mathbf{g}_j$ as $\mathbf{g}_j = \sum_{k=1}^{M} c_{kj}\mathbf{e}_k$. By Parseval's identity,

$$1 = \|\mathbf{g}_j\|^2 = \sum_{k=1}^{M} |c_{kj}|^2, \text{ for all } j \in \{1, \ldots, N\}. \tag{2}$$

Taking $\mathbf{f}$ to be $\mathbf{e}_l, l \in \{1, \ldots, M\}$ and using equation (1) we obtain

$$A = \sum_{j=1}^{N} |\langle \mathbf{e}_l, \mathbf{g}_j \rangle|^2 = \sum_{j=1}^{N} |c_{lj}^*|^2 = \sum_{j=1}^{N} |c_{lj}|^2. \tag{3}$$

From (2) we conclude that $\sum_{j=1}^{M}\sum_{k=1}^{N} |c_{kj}|^2 = N$; form (3) we at the same time have $\sum_{j=1}^{M}\sum_{k=1}^{N} |c_{kj}|^2 = MA$. Combining these two expressions together we conclude that $A = N/M$.

(b) For any $\mathbf{f} \in \mathbb{C}^M$ we have that

$$A\|\mathbf{f}\|^2 \le \sum_{j=1}^{N} |\langle \mathbf{f}, \mathbf{g}_j \rangle|^2 \le B\|\mathbf{f}\|^2. \tag{4}$$

Taking $\mathbf{f} = \mathbf{e}_l, l \in \{1, \ldots, M\}$ we obtain therefore that

$$\sum_{j=1}^{N} |\langle \mathbf{f}, \mathbf{g}_j \rangle|^2 = \sum_{j=1}^{N} |c_{lj}|^2. \tag{5}$$

Now, from (4) it follows that $AM\|\mathbf{f}\|^2 \leq M\sum_{j=1}^{N}\left|\langle \mathbf{f}, \mathbf{g}_j\rangle\right|^2 \leq MB\|\mathbf{f}\|^2$; from (5), on the other hand, we conclude $M\sum_{j=1}^{N}\left|\langle \mathbf{f}, \mathbf{g}_j\rangle\right|^2 = \sum_{l=1}^{M}\sum_{j=1}^{N}\left|\langle \mathbf{f}, \mathbf{g}_j\rangle\right|^2 = \sum_{l=1}^{M}\sum_{j=1}^{N}\left|c_{lj}\right|^2 = \sum_{j=1}^{N}\sum_{l=1}^{M}\left|c_{lj}\right|^2 = \sum_{j=1}^{N}\|\mathbf{g}_j\|^2 = N$.
We now see that

$$AM \leq N \leq BM$$

and so

$$A \leq \frac{N}{M} \leq B.$$

Problem 7: Frame bounds

If one removes elements from an ONB, and takes f as one of those elements, it is seen that $A = 0$ and $B = 1 < \infty$. If one adds countably infinite copies of a single basis element to the ONB to form a frame, it is seen that $A = 1 > 0$ and $B = +\infty$. Obviously, both sequences are not frames since $A = 0$ or $B = +\infty$ are not permissible frame bounds.

Problem 8: Tight frame as an orthogonal projection of an ONB

$\{\mathbb{P}\mathbf{e}_j\}_{j=1}^{N}$ spans $\mathcal{H}'$, but it is not a linearly independent set since $N > M$ and is therefore not a basis for $\mathcal{H}'$. Now we have for every $\mathbf{f} \in \mathcal{H}'$

$$\sum_{j=1}^{N}|\langle \mathbf{f}, \mathbb{P}\mathbf{e}_j\rangle|^2 = \sum_{j=1}^{N}|\langle \mathbb{P}^*\mathbf{f}, \mathbf{e}_j\rangle|^2 = \sum_{j=1}^{N}|\langle \mathbb{P}\mathbf{f}, \mathbf{e}_j\rangle|^2 = \|\mathbb{P}\mathbf{f}\|^2 = \|\mathbf{f}\|^2$$

where in the second equality we used the fact that orthogonal projections are self-adjoint and in the last equation the fact that for all $\mathbf{f} \in \mathcal{H}'$ we have $\mathbb{P}\mathbf{f} = \mathbf{f}$. We conclude that $\{\mathbb{P}\mathbf{e}_j\}_{j=1}^{N}$ is a tight frame for $\mathcal{H}'$ with frame bound 1.

Note: $\mathbb{P}\mathbf{f} = \mathbf{f}$ for all $\mathbf{f} \in \mathcal{H}'$ since $\mathbb{P}$ is onto $\mathcal{H}'$, i.e., there exists $\mathbf{h} \in \mathcal{H}$, not necessarily unique, such that $\mathbf{f} = \mathbb{P}\mathbf{h}$, and so $\mathbb{P}\mathbf{f} = \mathbb{P}\mathbb{P}\mathbf{h} = \mathbb{P}\mathbf{h} = \mathbf{f}$. ($\mathbb{P}$ is idempotent)