Mathematical methods for machine learning and signal processing SS 19

Problem set 1

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Problem 1: Overcomplete expansion in $\mathbb{R}^2$

Consider the following example discussed in class. Given the vectors

$$\mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}, \quad \mathbf{e}_3 = \mathbf{e}_1 - \mathbf{e}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$$

we found that any vector $\mathbf{x} \in \mathbb{R}^2$ can be represented as a linear combination

$$\mathbf{x} = \langle \mathbf{x}, \tilde{\mathbf{e}}_1 \rangle \mathbf{e}_1 + \langle \mathbf{x}, \tilde{\mathbf{e}}_2 \rangle \mathbf{e}_2 + \langle \mathbf{x}, \tilde{\mathbf{e}}_3 \rangle \mathbf{e}_3$$

where

$$\tilde{\mathbf{e}}_1 = 2\mathbf{e}_1, \quad \tilde{\mathbf{e}}_2 = -\mathbf{e}_3, \quad \tilde{\mathbf{e}}_3 = -\mathbf{e}_1.$$

1. Find another set of vectors $\tilde{\mathbf{e}}'_1, \tilde{\mathbf{e}}'_2, \tilde{\mathbf{e}}'_3$, neither of which is collinear to neither of the vectors $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$ and such that any vector $\mathbf{x} \in \mathbb{R}^2$ can be represented as

   $$\mathbf{x} = \langle \mathbf{x}, \tilde{\mathbf{e}}'_1 \rangle \mathbf{e}_1 + \langle \mathbf{x}, \tilde{\mathbf{e}}'_2 \rangle \mathbf{e}_2 + \langle \mathbf{x}, \tilde{\mathbf{e}}'_3 \rangle \mathbf{e}_3.$$

   Hint: Look for another right-inverse of the matrix

   $$\begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\end{bmatrix}.$$

2. Now consider the following example discussed in class. Given the vectors

   $$\mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}, \quad \tilde{\mathbf{e}}_1 = \begin{bmatrix}1\\-1\end{bmatrix}, \quad \tilde{\mathbf{e}}_2 = \begin{bmatrix}0\\\sqrt{2}\end{bmatrix}$$

   any vector $\mathbf{x} \in \mathbb{R}^2$ can be represented as

   $$\mathbf{x} = \langle \mathbf{x}, \mathbf{e}_1 \rangle \tilde{\mathbf{e}}_1 + \langle \mathbf{x}, \mathbf{e}_2 \rangle \tilde{\mathbf{e}}_2.$$

   Show that $\mathbf{x}$ can also be written as

   $$\mathbf{x} = \langle \mathbf{x}, \tilde{\mathbf{e}}_1 \rangle \mathbf{e}_1 + \langle \mathbf{x}, \tilde{\mathbf{e}}_2 \rangle \mathbf{e}_2.$$

   Is it possible in this case to find two other vectors $\mathbf{e}'_1, \mathbf{e}'_2$, neither of which is collinear to neither of the vectors $\mathbf{e}_1, \mathbf{e}_2$ such that

   $$\mathbf{x} = \langle \mathbf{x}, \mathbf{e}'_1 \rangle \tilde{\mathbf{e}}_1 + \langle \mathbf{x}, \mathbf{e}'_2 \rangle \tilde{\mathbf{e}}_2?$$

   If the answer is "yes", find these vectors. If the answer is "no", explain why this is not possible.

Problem 2: Equality in the Cauchy-Schwarz inequality

Prove the following statement: if the elements f and g of a Hilbert space satisfy $\left|\langle f, g\rangle\right| = \|f\|\|g\|$ and $g \neq 0$, then $f = cg$ for some $c \in \mathbb{C}$.

Hint: Assume $\|f\| = \|g\| = 1$ and $\langle f, g\rangle = 1$. Then $f - g$ and g are orthogonal, while $f = f - g + g$. Therefore, $\|f\|^2 = \|f - g\|^2 + \|g\|^2$.

Problem 3: Useful identities in a Hilbert Space

Prove that for any two elements f and g of a Hilbert space, the following equalities hold:

1. Parallelogram law
$$\|f+g\|^2 + \|f-g\|^2 = 2\left(\|f\|^2 + \|g\|^2\right)$$

2. Polarization identity
$$\langle f, g\rangle = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2 + \mathrm{i}\left[\|\frac{f}{\mathrm{i}} + g\|^2 - \|\frac{f}{\mathrm{i}} - g\|^2\right]\right)$$

Hint: Remember that in a Hilbert space the norm is defined through the inner product as
$$\|f\| = \sqrt{\langle f, f\rangle}.$$

Problem 4: Discrete Fourier transform (DFT) as a signal expansion

The DFT of an N-point signal $f[k]$, $k = 0, \ldots, N-1$, is defined as
$$\widehat{f}[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} f[k] e^{-\mathrm{i}2\pi \frac{n}{N} k}.$$

Find the corresponding inverse transform and show that the DFT can be interpreted as a frame expansion in $\mathbb{C}^N$. Compute the frame bounds. Is the underlying frame special?

Problem 5: Unitary transformation of a frame

Let $\{g_k\}_{k\in\mathcal{K}}$ be a frame for the Hilbert space $\mathcal{H}$ with frame bounds A and B. Let $\mathbb{U} : \mathcal{H} \to \mathcal{H}$ be a unitary operator. Show that the set $\{\mathbb{U}g_k\}_{k\in\mathcal{K}}$ is again a frame for $\mathcal{H}$ and compute the corresponding frame bounds.

Problem 6: Redundancy of a frame

Let $\{\mathbf{g}_k\}_{k=1}^N$ be a frame for $\mathbb{C}^M$ with $N > M$. Assume that the frame vectors are normalized such that $\|\mathbf{g}_k\| = 1$, $k = 1, \ldots, N$. The ratio N/M is called the redundancy of the frame.

1. Assume that $\{\mathbf{g}_k\}_{k=1}^N$ is a tight frame with frame bound A. Show that $A = N/M$.

2. Now assume that A and B are the frame bounds of $\{\mathbf{g}_k\}_{k=1}^N$. Show that $A \leq N/M \leq B$.

Problem 7: Frame bounds

Prove that the upper and the lower frame bound are unrelated: In an arbitrary Hilbert space $\mathcal{H}$ find a set $\{g_k\}_{k\in\mathcal{K}}$ with an upper frame bound $B < \infty$ but with the tightest lower frame bound $A = 0$; find another set $\{g_k\}_{k\in\mathcal{K}}$ with lower frame bound $A > 0$ but with the tightest upper frame bound $B = \infty$. Is it possible to find corresponding examples in the finite-dimensional space $\mathbb{C}^M$?

Problem 8: Tight frame as an orthogonal projection of an ONB

Let $\{\mathbf{e}_k\}_{k=1}^N$ be an ONB for an N-dimensional Hilbert space $\mathcal{H}$. For $M < N$, let $\mathcal{H}'$ be an M-dimensional subspace of $\mathcal{H}$. Let $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto $\mathcal{H}'$. Show that $\{\mathbb{P}\mathbf{e}_k\}_{k=1}^N$ is a tight frame for $\mathcal{H}'$. Find the corresponding frame bound.